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CHAPTER I

On Bicomplex Lorentz Sequence Space

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1.Introduction

The widely known appendage of the complex number field to the four-dimensional field is by a work titled "Algebra on Quaternions or a new imaginary system" by W.R. Hamilton in 1844. As an idea, Quaternions emerged by take into account three imaginary units (i, j, k), where $k = ij$, which are non-commutative. The importance of the speculation of quaternions is that it creates a field in which entire known operations can be carried out. Loss of commutativity can be considered as a deficiency. Although from a

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fully algebraic point of view the deficiency of commutativity is not such a horrible problem, in some studies it poses many hardship, for example, when we try to expand the fertile speculation of holomorphic functions of a complex variable to quaternions.

Therefore, it is not illogical to think whether a four-dimensional algebra that includes $\mathbb{C}$ as the lower algebra could be included in a way that conserves the commutative property. It is not surprising that this is done by taking into account two imaginary units i, j by publicity $k = ij$ (even as in the quaternionic case), but now majestic $ij = ji$. This converts k to the hyperbolic imaginary unit, i.e. something like $k^2 = 1$. As far as is known, J. Cockle wrote a series of articles in 1848 regarding these new algebra studies. Cockle's work was absolutely encouraged by Hamilton's and he was the first to use tessarin-algebra to insulate the hyperbolic trigonometric series as components of the exponential series. Not surprisingly, Cockle instantly noticed that there was a price to be paid for commutativity in four dimensions, and the compensation was the asset of zero-divisors.

Only in 1892, inspired by the study of Hamilton and Clifford, did the mathematician Corrado Segre introduce what he named the algebra of bicomplex numbers, the equivalent of tessarine algebra. In his individual papers, Segre recognized that the elements $((1+ij))/2$ and $((1-ij))/2$ are idempotent and play a head role in even complex number speculation . After Segre, several other mathematicians, notably (Spampinato, 1935, 1936) and (Scorza Dragoni, 1934), advanced the first basics of a speculation of functions on even complex numbers.

The subsequent grand breakthrough in the work of bicomplex analysis was the work of J.D. Riley in 1953, in which he further advanced the speculation of functions of bicomplex variables. However, the most significant contribution was undoubtedly the work of G. B. Price (Price, 1991); here the speculation of holomorphic functions of two complex variables (as well as very complex variables) was fully developed. Until this monograph, the work of G. B. Price is accepted the foundational study of this speculation.

However, recently there has been a significant revival of interest in the work of holomorphic functions on one and several pairs of complex variables, as well as in improvement functional analysis on spaces having a build consisting of modules on two rings of complex numbers. B. Sağır one of those who carried out these studies has carried out many studies on $\mathbb{BC}$.

We now explain the content of this study. First of all, the bicomplex number $\mathbb{BC}$ is introduced and its features is explained with examples. Hyperbolic numbers, which are a subset of the bicomplex set, will be mentioned. A hyperbolic-valued norm is defined. The measure space is introduced and accordingly the Distribution, Decreasing rearrangement and Average function definitions of a function is made. Without changing the basis of these definitions, by using the set of Hyperbolic numbers and the Hyperbolic valued norm, the $\mathbb{D}$ -Distribution, $\mathbb{D}$ -Decreasing rearrangement and $\mathbb{D}$ -Average functions of a sequence with Bicomplex terms are created and the properties of these functions are examined.

Definition of Bicomplex Numbers

The members of set represented by $\mathbb{BC} = \{z_1 + jz_2 : z_1, z_2 \in \mathbb{C}\}$ is called bicomplex numbers, where $\mathbb{C}$ is the set of complex numbers with the virtual unit i, and also where i and $i \neq j$ are commuting virtual units, i.e., $ij = ji = k$, $i^2 = j^2 = -1$ and

$$k^2 = (ij)^2 = (ij)(ij) = i(ji)j = i(ij)j = (ii)(jj) = i^2 j^2 = 1.$$

So bicomplex numbers are "complex numbers with complex coefficients", which express the name of bicomplex, and in what follows we will attempt to emphasize the similarities between the feature of complex and bicomplex numbers.

Any two elements in the set of bicomplex numbers can be added and multiplied. Let $u = u_1 + ju_2$ and $v = v_1 + jv_2$ are two bicomplex numbers, the formulas for the summation and the product of two bicomplex numbers are given by

$$u + v = (u_1 + v_1) + j(u_2 + v_2) \tag{1.1}$$

and

$$u \cdot v = (u_1 v_1 - u_2 v_2) + j(u_1 v_2 + u_2 v_1) \tag{1.2}$$

respectively. According to operations (1.1) and (1.2) described above, the set $\mathbb{BC}$ is a commutative and unitary ring with $1_{\mathbb{BC}} = 1 + j \cdot 0$.

Example 1. Investigate whether the bicomplex number $u = (2 - i) + j(1 + 2i)$ is invertible in the $\mathbb{BC}$ space.

Solution. Assume that

$$u^{-1} = v = v_1 + jv_2 = (x_1 + iy_1) + j(x_2 + iy_2)$$

with $(x_1, y_1, x_2, y_2) \in \mathbb{R}^4$. Then

$$\begin{aligned} u \cdot v &= \big((2 - i) + j(1 + 2i)\big) \cdot \big((x_1 + iy_1) + j(x_2 + iy_2)\big) \\ &= (2x_1 + y_1 - x_2 + 2y_2) + i(-x_1 + 2y_1 - 2x_2 - y_2) \\ &\quad + j(x_1 - 2y_1 + 2x_2 + y_2) + ij(2x_1 + y_1 - x_2 + 2y_2) \\ &= 1 + j \cdot 0 \end{aligned}$$

and hence a system of equations

$$\begin{aligned} 2x_1 + y_1 - x_2 + 2y_2 &= 1 \\ -x_1 + 2y_1 - 2x_2 - y_2 &= 0 \\ x_1 - 2y_1 + 2x_2 + y_2 &= 0 \\ 2x_1 + y_1 - x_2 + 2y_2 &= 0 \end{aligned} \tag{1.3}$$

is obtained, if the coefficients of the mutual imaginary components in this equation are equalized. Since the system of equations (1.3) has no solution, $u = (2 - i) + j(1 + 2i)$ has no inverse in $\mathbb{BC}$. Then, since every element of $\mathbb{BC}$ is not invertible with respect to multiplication, we can say that $\mathbb{BC}$ is not a field.

In addition, when $z_2 = 0$ in $z = z_1 + jz_2$, that is, $z = z_1$, the set of these numbers is represented by $\mathbb{C}(i)$. If the z_1 and z_2 coefficients are real numbers, that is, $z = x_1 + jx_2$ with $x_1, x_2 \in \mathbb{R}$, the set of those numbers is represented by $\mathbb{C}(j)$. $\mathbb{C}(i)$ and $\mathbb{C}(j)$ are isomorphic fields but coexisting internal of $\mathbb{BC}$ they are distinct. We will see many times in what follows that there is a specific asymmetry in their attitude.

Definition 1. The set of hyperbolic numbers is described by

$$\mathbb{D} = \{x + ky : x, y \in \mathbb{R}, k = ij\},$$

where k is a hyperbolic virtual unit, i.e., $k^2 = 1$. In the studies conducted in the current literature, hyperbolic numbers are sometimes called duplex, double or bireal numbers. The following subsets $\mathbb{D}^+$ and $\mathbb{D}^+\backslash\{0\}$ of $\mathbb{D}$ are called non-negative and positive hyperbolic numbers, respectively:

$$\mathbb{D}^+ = \{x + ky : x^2 - y^2 \geq 0, x^2 \geq 0\} \tag{1.4}$$

$$\mathbb{D}^+\backslash\{0\} = \{x + ky : x^2 - y^2 \geq 0, x^2 > 0\}. \tag{1.5}$$

Similarly, non-negative and negative hyperbolic numbers are defined as follows:

$$\mathbb{D}^- = \{x + ky : x^2 - y^2 \geq 0, x^2 \leq 0\} \tag{1.6}$$

$$\mathbb{D}^-\backslash\{0\} = \{x + ky : x^2 - y^2 \geq 0, x^2 < 0\}. \tag{1.7}$$

Idempotent Representations of Bicomplex Numbers

Consider the bicomplex numbers $e_1 = \frac{1+ij}{2}$ and $e_2 = \frac{1-ij}{2}$. It can be easily seen that $e_1 \cdot e_2 = e_2 \cdot e_1 = 0$. There are also equations $(e_1)^n = e_1$, $(e_2)^n = e_2$ with $n \geq 2$. For any bicomplex number $u = u_1 + ju_2 \in \mathbb{BC}$, we have

$$u = \frac{u_1 + iu_2 + u_1 - iu_2}{2} + j\frac{u_2 + iu_1 + u_2 - iu_1}{2}$$

$$= \frac{u_1 - iu_2}{2} + \frac{u_1 + iu_2}{2} + j\left(i\frac{u_1 - iu_2}{2} - i\frac{u_1 + iu_2}{2}\right)$$

$$= \frac{u_1 - iu_2}{2}(1 + ij) + \frac{u_1 + iu_2}{2}(1 + ij)$$

$$= (u_1 - iu_2)e_1 + (u_1 + iu_2)e_2 = \delta_1 e_1 + \delta_2 e_2 \qquad (1.8)$$

with $\delta_1 = (u_1 - iu_2)$ and $\delta_2 = (u_1 + iu_2)$ in $\mathbb{C}(i)$. This equality (1.8) is named the $\mathbb{C}(i)$-idempotent representation of the bicomplex number u.

Similarly, along with the coefficients in $\mathbb{C}(j)$, there is also a representation of the bicomplex number u with respect to e_1 and e_2.

As a result, any bicomplex number has an idempotent representation with its coefficients in any of $\mathbb{C}(i)$ or $\mathbb{C}(j)$, that is, $u = \delta_1 e_1 + \delta_2 e_2 = \rho_1 e_1 + \rho_2 e_2$ where $\delta_1, \delta_2 \in \mathbb{C}(i)$ and $\rho_1, \rho_2 \in \mathbb{C}(j)$.

A Partial Order on The Set of Hyperbolic Numbers

Let u and v be two elements of the set $\mathbb{D}^+$. If $u - v \in \mathbb{D}^+$, that is, $u - v$ is a non-negative hyperbolic number, then we write $u \succcurlyeq v$ or $v \preccurlyeq u$, and also u is said to be $\mathbb{D}$-greater or $\mathbb{D}$-equal than v, or that v is $\mathbb{D}$-less or $\mathbb{D}$-equal than u.

If $u = u_1 e_1 + u_2 e_2$ and $v = v_1 e_1 + v_2 e_2$, with real numbers u_1, u_2, v_1 and v_2, we have

$$v \preccurlyeq u \Leftrightarrow v_1 \leq u_1 \ \wedge \ v_2 \leq u_2. \qquad (1.9)$$

Also u is a (absolutely) positive hyperbolic number, then it is reversed and its inverse is also positive. Additionally, if $u \succ 0$ and $u \prec v$, then $v^{-1} \succ 0$ and $v^{-1} \prec u^{-1}$ (Luna-Elizarrarás, M.E. & et al., 2015).

The Hyperbolic-Valued $\mathbb{D}$-Norm on $\mathbb{BC}$

A function $|.|_k$ defined from $\mathbb{BC}$ to $\mathbb{D}^+$ with $|u|_k = |u_1|e_1 + |u_2|e_2$ for each $u = u_1 e_1 + u_2 e_2 \in \mathbb{BC}$ provides the following properties.

$N_1)$ Since $|u_1| \geq 0$ and $|u_2| \geq 0$ for a $u = (u_1 e_1 + u_2 e_2)$ in $\mathbb{BC}$, we have $|u|_k = |u_1|e_1 + |u_2|e_2 \succcurlyeq 0e_1 + 0e_2 = 0$.

N_2) Let $|u|_k = |u_1|e_1 + |u_2|e_2 = 0 = 0e_1 + 0e_2$ for any $u = (u_1e_1 + u_2e_2) \in \mathbb{BC}$. Hence $|u_1| = 0$ ve $|u_2| = 0$, namely,

$u = 0e_1 + 0e_2 = 0$. It is clear that $|u|_k = 0$ when $u = 0$.

N_3) Let $u = (u_1e_1 + u_2e_2)$ in $\mathbb{BC}$ and λ in $\mathbb{D}$. Then

$$\begin{aligned}|\lambda u|_k &= |(\lambda_1e_1 + \lambda_2e_2)(u_1e_1 + u_2e_2)|_k \\ &= |(\lambda_1u_1)e_1 + (\lambda_2u_2)e_2|_k \\ &= |\lambda_1u_1|e_1 + |\lambda_2u_2|e_2 \\ &= |\lambda_1||u_1|e_1 + |\lambda_2||u_2|e_2 \\ &= (|\lambda_1|e_1 + |\lambda_2|e_2)(|u_1|e_1 + |u_2|e_2) \\ &= |\lambda|_k|u|_k.\end{aligned}$$

N_4) Let u and v with $u = (u_1e_1 + u_2e_2)$ and $v = (v_1e_1 + v_2e_2)$ be two elements in $\mathbb{BC}$. Then

$$\begin{aligned}|u+v|_k &= |(u_1e_1 + u_2e_2) + (v_1e_1 + v_2e_2)|_k \\ &= |(u_1 + v_1)e_1 + (u_2 + v_2)e_2|_k \\ &= |u_1 + v_1|e_1 + |u_2 + v_2|e_2 \\ &\preccurlyeq (|u_1| + |v_1|)e_1 + (|u_2| + |v_2|)e_2 \\ &= (|u_1|e_1 + |u_2|e_2) + (|v_1|e_1 + |v_2|e_2) \\ &= |u|_k + |v|_k.\end{aligned}$$

As a result of the conditions N_1), N_2), N_3) and N_4), it is seen that $|\cdot|_k$ is a $\mathbb{D}$-norm. This norm is named the hyperbolic-valued norm.

Distribution Functions

A collection $\mathcal{G}$ of subsets of a set G is named a $\sigma -$algebra if it supply the following circumstances:

i. $G \in \mathcal{G}$.

ii. $S \in \mathcal{G}$ then $G \backslash S \in \mathcal{G}$.

iii. If $(I_k)_k \subseteq \mathcal{G}$ is a sequence, then $\bigcup_{k=1}^{\infty} I_k \in \mathcal{G}$.

The couple $(G, \mathcal{G})$ is named a measurable space and each element of $\mathcal{G}$ is named a measurable set (Halmos, 1978).

Let $\mathcal{G}$ be a σ-algebra. A function $\mu: \mathcal{G} \rightarrow [0, \infty)$ is named a measure, if it supply the following circumstances:

i. $\mu(\emptyset) = 0$.

ii. $\mu(\bigcup_{k=1}^{\infty} I_k) = \sum_{k=1}^{\infty} \mu(I_k)$ for any sequence $(I_k)_k$ of pairwise disjoint sets from $\mathcal{G}$, that is, $I_k \cap I_j = \emptyset$ for $j \neq k$.

Additionally, the trinity $(G, \mathcal{G}, \mu)$ is said a measure space.

Definition 2. Let $(G, \mathcal{G}, \mu)$ be a measure space and $\mathfrak{M}(G, \mathcal{G})$ be the set of all measurable complex-valued functions on G. The distribution function D_g of a function g in $\mathfrak{M}(G, \mathcal{G})$ is given by

$$D_g(\lambda) = \mu\{x \in G: |g(x)| > \lambda \geq 0\} \tag{1.10}$$

(Castillo & Rafeiro, 2015).

Decreasing Rearrangement

Definition 3. Let a $g \in \mathfrak{M}(G, \mathcal{G})$ be given. The function determined as

$$g^*(t) = inf\{\lambda \geq 0: D_g(\lambda) \leq t\} = sup\{\lambda \geq 0: D_g(\lambda) > t\} \tag{1.11}$$

is described the decreasing rearrangement function of g. According to this definition, it is clear that the function g^* is defined from $[0, \infty]$ to $[0, \infty]$ (Eryılmaz& Işık, 2019).

Definition 4. Let a $g \in \mathfrak{M}(G, \mathcal{G})$ be given. The average function of g^*, which we will denote by g^{**}, is defined by

$$g^{**}(m) = \frac{1}{m} \int_0^m g^*(t)\, dt, m > 0 \tag{1.12}$$

(Eryılmaz& Işık, 2019).

2. $\mathbb{D}$-Distribution and $\mathbb{D}$-Decreasing Rearrangement Functions of Sequences with Bicomplex Terms

Let $\mathcal{W}_{\mathbb{BC}}$ be the set of sequences with all bicomplex terms and $\mathcal{G}$ be the power set of $\mathbb{N}$, namely $\mathcal{G} = 2^{\mathbb{N}}$, and μ be counting measure on $\mathcal{G}$.

$\mathbb{D}$-Distribution function

In the definition of the $\mathbb{D}$-distribution function, a sequence with bicomplex terms instead of the measurable function and the non-negative hyperbolic number instead of the non-negative real number in definition of general distribution function, given in (1.10), is used.

Definition 5. Let $u = \big(u(n)\big)_n$ with $u(n) = u_1(n)e_1 + u_2(n)e_2$ be a arbitrary sequence in $\mathbb{BC}$ and a number $\lambda_1 e_1 + \lambda_2 e_2 = \lambda \in \mathbb{D}^+$ be given. The $\mathbb{D}$-distribution function $\mathcal{D}_u$ of u is defined by

$$\mathcal{D}_u(\lambda) = D_{u_1}(\lambda_1)e_1 + D_{u_2}(\lambda_2)e_2. \qquad (2.1)$$

Definition 6. A function $\hbar$ defined from $\mathbb{D}^+$ to $\mathbb{D}^+$, is called a $\mathbb{D}$-decreasing function, if there is a $\mathbb{D}$-inequality $\hbar(\beta) \preccurlyeq \hbar(\alpha)$, whenever $\alpha \prec \beta$.

Lemma 1. Let λ be an element in $\mathbb{D}$ with $\lambda = \lambda_1 e_1 + \lambda_2 e_2$ and δ be an element in $\mathbb{D}$ with $\delta = \delta_1 e_1 + \delta_2 e_2$, $\delta_1 \neq 0$ and $\delta_2 \neq 0$. There exist the equality

$$\frac{\lambda}{|\delta|_k} = \frac{\lambda_1}{|\delta_1|}e_1 + \frac{\lambda_2}{|\delta_2|}e_2.$$

Proof. We have

$$\frac{\lambda}{|\delta|_k} = \frac{\lambda_1 e_1 + \lambda_2 e_2}{|\delta_1|e_1 + |\delta_2|e_2} = \frac{\lambda_1 e_1}{|\delta_1|e_1 + |\delta_2|e_2} + \frac{\lambda_2 e_2}{|\delta_1|e_1 + |\delta_2|e_2}$$

$$= \frac{\lambda_1 e_1 e_1}{(|\delta_1|e_1 + |\delta_2|e_2)e_1} + \frac{\lambda_2 e_2 e_2}{(|\delta_1|e_1 + |\delta_2|e_2)e_2}$$

$$= \frac{\lambda_1 e_1 e_1}{|\delta_1| e_1} + \frac{\lambda_2 e_2 e_2}{|\delta_2| e_2} = \frac{\lambda_1}{|\delta_1|} e_1 + \frac{\lambda_2}{|\delta_2|} e_2.$$

Theorem 1. Let $u = (u(n))$ and $v = (v(n))$ be two sequences in $\mathcal{W}_{\mathbb{BC}}$ and λ, δ and c be in $\mathbb{D}^+$. Then, the following features are satisfied:

a) $\mathbb{D}$-distribution function is $\mathbb{D}$-decreasing.

b) If $|v(n)|_k \preccurlyeq |u(n)|_k$ for all $n \geq 1$, then $\mathcal{D}_v(\lambda) \preccurlyeq \mathcal{D}_u(\lambda)$.

c) $\mathcal{D}_{cu}(\lambda) = \mathcal{D}_u\left(\frac{\lambda}{|c|_k}\right)$ for all $c \in \mathbb{D}^+$, where $c = c_1 e_1 + c_2 e_2$ and $c_1 \neq 0 \neq c_2$.

d) $\mathcal{D}_{u+v}(\lambda + \delta) \preccurlyeq \mathcal{D}_u(\lambda) + \mathcal{D}_v(\delta)$.

e) $\mathcal{D}_{u \cdot v}(\lambda \cdot \delta) \preccurlyeq \mathcal{D}_u(\lambda) + \mathcal{D}_v(\delta)$.

Proof. a) Assume that $\lambda = \lambda_1 e_1 + \lambda_2 e_2$, $\delta = \delta_1 e_1 + \delta_2 e_2$ and $\delta \prec \lambda$. Since distribution functions D_{u_1} and D_{u_2} are decreasing (Castillo & Rafeiro, 2015), obviously

$$\mathcal{D}_u(\lambda) = D_{u_1}(\lambda_1)e_1 + D_{u_2}(\lambda_2)e_2$$
$$\preccurlyeq D_{u_1}(\delta_1)e_1 + D_{u_2}(\delta_2)e_2 = \mathcal{D}_u(\delta).$$

Hence $\mathcal{D}_u$ is $\mathbb{D}$-decreasing function.

b) Assume that $|v(n)|_k \preccurlyeq |u(n)|_k$ for all $n \geq 1$. Then we can write

$$|v_1(n)|e_1 + |v_2(n)|e_2 \preccurlyeq |u_1(n)|e_1 + |u_2(n)|e_2$$
$$\Leftrightarrow |v_1(n)| \leq |u_1(n)| \text{ and } |v_2(n)| \leq |u_2(n)|.$$

Thus

$$\{n \in \mathbb{N}: |v_m(n)| > \lambda_1\} \subseteq \{n \in \mathbb{N}: |u_m(n)| > \lambda_1\}, m = 1,2,$$

and therefore $D_{v_1}(\lambda_1) \leq D_{u_1}(\lambda_1)$ and $D_{v_2}(\lambda_2) \leq D_{u_2}(\lambda_2)$. This shows that $\mathcal{D}_v(\lambda) \preccurlyeq \mathcal{D}_u(\lambda)$.

c) According to the definition of $\mathbb{D}$-distribution function,

$$\mathcal{D}_{cu}(\lambda) = D_{c_1u_1}(\lambda_1)e_1 + D_{c_2u_2}(\lambda_2)e_2$$

is written. We can easily see that

$$\mu\{n \in \mathbb{N}: |c_m u_m(n)| > \lambda_1\} = \mu\left\{n \in \mathbb{N}: |u_m(n)| > \frac{\lambda_1}{|c_m|}\right\}$$

$$= D_{u_m}\left(\frac{\lambda_1}{|c_m|}\right), m = 1,2.$$

Then

$$D_{c_1u_1}(\lambda_1)e_1 + D_{c_2u_2}(\lambda_2)e_2 = D_{u_1}\left(\frac{\lambda_1}{|c_1|}\right)e_1 + D_{u_2}\left(\frac{\lambda_2}{|c_2|}\right)e_2$$

and hence

$$\mathcal{D}_{cu}(\lambda) = \mathcal{D}_u\left(\frac{\lambda}{|c|_k}\right).$$

d) Considering the distribution functions $D_{(u_m+v_m)}(\lambda_m + \delta_m)$ with $m = 1,2$; we have

$$\{n \in \mathbb{N}: \lambda_m + \delta_m < |u_m(n) + v_m(n)|\} \subset$$
$$\{n \in \mathbb{N}: \lambda_m < |u_m(n)|\} \cup \{n \in \mathbb{N}: \delta_m < |v_m(n)|\}$$

and hence

$$D_{(u_m+v_m)}(\lambda_m + \delta_m) \leq D_{u_m}(\lambda_m) + D_{v_m}(\delta_m).$$

Using these inequalities, we obtain $\mathbb{D}$-inequality

$$\mathcal{D}_{u+v}(\lambda + \delta) = D_{(u_1+v_1)}(\lambda_1 + \delta_1)e_1 + D_{(u_2+v_2)}(\lambda_2 + \delta_2)e_2$$
$$\preccurlyeq \left\{D_{u_1}(\lambda_1) + D_{v_1}(\delta_1)\right\}e_1 + \left\{D_{u_2}(\lambda_2) + D_{v_2}(\delta_2)\right\}e_2$$
$$= \left\{D_{u_1}(\lambda_1)e_1 + D_{u_2}(\lambda_2)e_2\right\} + \left\{D_{v_1}(\delta_1)e_1 + D_{v_2}(\delta_2)e_2\right\}$$

finally

$$\mathcal{D}_{u+v}(\lambda + \delta) \preccurlyeq \mathcal{D}_u(\lambda) + \mathcal{D}_v(\delta). \qquad (2.2)$$

e) Using the definitions of the distribution functions

$D_{(u_m \cdot v_m)}(\lambda_m \cdot \delta_m)$ with $m = 1,2$; we have

$$\{n \in \mathbb{N}: \lambda_m \cdot \delta_m < |u_m(n) \cdot v_m(n)|\}$$
$$\subset \{n \in \mathbb{N}: \lambda_m < |u_m(n)|\} \cup \{n \in \mathbb{N}: \delta_m < |v_m(n)|\}$$

and hence

$$D_{(u_m \cdot v_m)}(\lambda_m \cdot \delta_m) \leq D_{u_m}(\lambda_m) + D_{v_m}(\delta_m)$$

Also the inequalities above allow us to write

$$\mathcal{D}_{u \cdot v}(\lambda \cdot \delta) = D_{(u_1 \cdot v_1)}(\lambda_1 \cdot \delta_1)e_1 + D_{(u_2 \cdot v_2)}(\lambda_2 \cdot \delta_2)e_2$$
$$\preccurlyeq \{D_{u_1}(\lambda_1) + D_{v_1}(\delta_1)\}e_1 + \{D_{u_2}(\lambda_2) + D_{v_2}(\delta_2)\}e_2$$
$$= \{D_{u_1}(\lambda_1)e_1 + D_{u_2}(\lambda_2)e_2\} + \{D_{v_1}(\delta_1)e_1 + D_{v_2}(\delta_2)e_2\}$$

and so

$$\mathcal{D}_{u \cdot v}(\lambda \cdot \delta) \preccurlyeq \mathcal{D}_u(\lambda) + \mathcal{D}_v(\delta). \qquad (2.3)$$

$\mathbb{D}$-Decreasing Rearrangement Function

Definition 7. If there is $\rho \in \mathbb{D}$ such that $g \preccurlyeq \rho (\rho \preccurlyeq g)$ for all $g \in G$, then it is said that a subset $G \subset \mathbb{D}$ is $\mathbb{D}$-frontiered from above(lower). This number $\rho \in \mathbb{D}$ is called a $\mathbb{D}$-top($\mathbb{D}$-inferior) frontier of G.

If $G \subset \mathbb{D}$ is a $\mathbb{D}$-frontiered set from upstairs, then we describe the its $\mathbb{D}$-supremum showed by $sup_{\mathbb{D}} G$, the smallest top frontier of G, and its $\mathbb{D}$-infimum showed by $inf_{\mathbb{D}} G$, the hugest inferior frontier of G. The “least” top frontier here means that $sup_{\mathbb{D}} G \preccurlyeq \rho$ for any $\mathbb{D}$-top frontier ρ even if not all of the $\mathbb{D}$-top frontiers are comparable. Likewise, the meaning of the “hugest” inferior frontier can be evaluate. Surely, every non-empty set of hyperbolic numbers which is $\mathbb{D}$-frontier from upstairs has its $\mathbb{D}$-supremum, and if it is $\mathbb{D}$-frontier from inferior, then it has a $\mathbb{D}$-infimum. Given a set $G \subset \mathbb{D}$, think the sets

$$G_1 = \{g_1: g_1 e_1 + g_2 e_2 \in G\} \text{ and } G_2 = \{g_2: g_1 e_1 + g_2 e_2 \in G\}.$$

If G is a $\mathbb{D}$-frontiered set from upstairs, then the $sup_{\mathbb{D}} G$ can be calculated by the formula

$$sup_{\mathbb{D}}G = supG_1e_1 + supG_2e_2. \tag{2.4}$$

If G is a $\mathbb{D}$-frontiered set from below, then the $inf_{\mathbb{D}}G$ can be calculated by the formula

$$inf_{\mathbb{D}}G = inf\,G_1e_1 + inf\,G_2e_2. \tag{2.5}$$

If G and H are two $\mathbb{D}$-frontiered set from inferior, then so is $G + H$ and

$$inf_{\mathbb{D}}(G + H) = inf_{\mathbb{D}}G + inf_{\mathbb{D}}H \tag{2.6}$$

holds.

If two subsets $G \subset \mathbb{D}^+$ and $H \subset \mathbb{D}^+$ are $\mathbb{D}$-frontiered from inferior, then so is $G \cdot H$ and

$$inf_{\mathbb{D}}(G \cdot H) = inf_{\mathbb{D}}G \cdot inf_{\mathbb{D}}H \tag{2.7}$$

holds.

For the top $\mathbb{D}$-frontiered subsets of $\mathbb{D}$, the equations (2.6) to (2.7) are still true when $sup_{\mathbb{D}}$ is written instead of $inf_{\mathbb{D}}$ (Luna-Elizarrarás, M.E. & et al., 2015).

Definition 8. The $\mathbb{D}$-decreasing rearrangement of a sequence $\mathcal{u} = (\mathcal{u}(n))$ in $\mathcal{W}_{\mathbb{BC}}$ is a function $\mathcal{u}^*$ defined by

$$\mathcal{u}^*(t) = inf_{\mathbb{D}}\{\lambda \in \mathbb{D}^+ : \mathcal{D}_{\mathcal{u}}(\lambda) \preccurlyeq t\}$$

from $\mathbb{D}^+$ to $\mathbb{D}^+$. Since $D_{\mathcal{u}_i}$ with $i = 1,2$ is decreasing,

$$inf\{\lambda_1 \geq 0: D_{\mathcal{u}_1}(\lambda_1) \leq D_{\mathcal{u}_2}(\beta_1)\} = \beta_1$$

and

$$inf\{\lambda_2 \geq 0: D_{\mathcal{u}_2}(\lambda_2) \leq D_{\mathcal{u}_2}(\beta_2)\} = \beta_2$$

holds. Therefore,

$$\mathcal{u}^*\left(\mathcal{D}_{\mathcal{u}}(\beta)\right) = inf_{\mathbb{D}}\{\lambda \in \mathbb{D}^+ : \mathcal{D}_{\mathcal{u}}(\lambda) \preccurlyeq \mathcal{D}_{\mathcal{u}}(\beta)\}$$

$$= inf\{\lambda_1 \geq 0: D_{\mathcal{u}_1}(\lambda_1) \leq D_{\mathcal{u}_2}(\beta_1)\}e_1$$

$$+inf\{\lambda_2 \geq 0: D_{u_2}(\lambda_2) \leq D_{u_2}(\beta_2)\}e_2 = \beta,$$

$\mathcal{u}^*$ is the left inverse of $\mathcal{D}_u$.

Morever, let $u(n) = u_1(n)e_1 + u_2(n)e_2$, $\lambda = (\lambda_1 e_1 + \lambda_2 e_2)$ and $t = (t_1 e_1 + t_2 e_2)$. Then

$$u^*(t) = inf_{\mathbb{D}}\{\lambda \in \mathbb{D}^+ : \mathcal{D}_u(\lambda) \preccurlyeq t\}$$

$$= inf_{\mathbb{D}}\{\lambda \in \mathbb{D}^+ : D_{u_1}(\lambda_1)e_1 + D_{u_2}(\lambda_2)e_2 \preccurlyeq t_1 e_1 + t_2 e_2\}$$

$$= inf\{\lambda_1 : D_{u_1}(\lambda_1) \leq t_1\}e_1 + inf\{\lambda_2 : D_{u_2}(\lambda_2) \leq t_2\}e_2$$

and so

$$u^*(t) = u_1{}^*(t_1)e_1 + u_2{}^*(t_2)e_2. \quad (2.8)$$

Theorem 2. Let $u = (u(n))$ be a sequence in $\mathcal{W}_{\mathbb{BC}}$ and $\lambda = (\lambda_1 e_1 + \lambda_2 e_2)$, $t = (t_1 e_1 + t_2 e_2)$ be two elements in $\mathbb{D}^+$. The $\mathbb{D}$-decreasing rearrangement function of u has the following properties:

a) u^* is $\mathbb{D}$-decreasing.

b) $u^*(t) \succ \lambda$ necessary and sufficient condition $\mathcal{D}_u(\lambda) \succ t$.

c) $(\kappa u)^*(t) = |\kappa|_k u^*(t)$, $\kappa \in \mathbb{D}^+$.

d) Let $(u(n))$ and $(v(n))$ be two sequences in $\mathcal{W}_{\mathbb{BC}}$. If $|u(n)|_k \prec |v(n)|_k$ for every $n = 1,2, \ldots$ then $u^*(t) \preccurlyeq v^*(t)$.

e) $[(|u(n)|_k)^p]^* = \left(u^*(n)\right)^p$, $p \geq 1$.

Proof.

a) Let $0 \preccurlyeq t \prec k$, then $t_1 \leq k_1 \wedge t_2 \leq k_2$.

This shows that

$$\{\lambda_1 \geq 0: D_{u_1}(\lambda_1) \leq t_1\} \subset \{\lambda_1 \geq 0: D_{u_1}(\lambda_1) \leq k_1\}$$

and

$$\{\lambda_2 \geq 0: D_{u_2}(\lambda_2) \leq t_2\} \subset \{\lambda_2 \geq 0: D_{u_2}(\lambda_2) \leq k_2\}$$

therefore,

$$\{\lambda \in \mathbb{D}^{+}: \mathcal{D}_{u}(\lambda) \preccurlyeq t\} \subset \{\lambda \in \mathbb{D}^{+}: \mathcal{D}_{u}(\lambda) \preccurlyeq k\}$$

obviously

$$inf_{\mathbb{D}}\{\lambda \in \mathbb{D}^{+}: \mathcal{D}_{u}(\lambda) \preccurlyeq k\} \preccurlyeq inf_{\mathbb{D}}\{\lambda \in \mathbb{D}^{+}: \mathcal{D}_{u}(\lambda) \preccurlyeq t\}$$

that is,

$$u^{*}(k) \preccurlyeq u^{*}(t).$$

b) Let $\lambda \prec u^{*}(t)$. Then

$$\lambda \prec inf_{\mathbb{D}}\{\alpha \in \mathbb{D}^{+}: \mathcal{D}_{u}(\alpha) \preccurlyeq t\}$$

namely

$$\begin{aligned} &\lambda_1 e_1 + \lambda_2 e_2 \\ &\quad \prec inf_{\mathbb{D}}\left\{\alpha \in \mathbb{D}^{+}: D_{u_1}(\alpha_1)e_1 + D_{u_2}(\alpha_2)e_2 \preccurlyeq t_1 e_1 + t_2 e_2\right\} \\ &\quad = inf\left\{\alpha_1: D_{u_1}(\alpha_1) \le t_1\right\}e_1 + inf\left\{\alpha_2: D_{u_2}(\alpha_2) \le t_2\right\}e_2. \end{aligned}$$

Thus

$$\lambda_1 < inf\left\{\alpha_1: D_{u_1}(\alpha_1) \le t_1\right\} \text{ and } \lambda_2 < inf\left\{\alpha_2: D_{u_2}(\alpha_2) \le t_2\right\}$$

and so $D_{u_1}(\lambda_1) > t_1$, $D_{u_2}(\lambda_2) > t_2$. Obviously $\mathcal{D}_{u}(\lambda) \succ t$.

Conversely, let $t \prec \mathcal{D}_{u}(\lambda)$. Suppose that $u^{*}(t) \preccurlyeq \lambda$. Since $\mathcal{D}_{u}$ is $\mathbb{D}$-decreasing, we have $\mathcal{D}_{u}(\lambda) \preccurlyeq \mathcal{D}_{u}\left(u^{*}(t)\right) \preccurlyeq t$ contradicting our hypothesis.

c) By the Theorem 1, we write

$$\begin{aligned} (\kappa u)^{*}(t) &= inf_{\mathbb{D}}\{\alpha \in \mathbb{D}^{+}: \mathcal{D}_{\kappa u}(\alpha) \preccurlyeq t\} \\ &= inf_{\mathbb{D}}\left\{\alpha \in \mathbb{D}^{+}: \mathcal{D}_{u}\left(\frac{\alpha}{|\kappa|_k}\right) \preccurlyeq t\right\}. \end{aligned}$$

If it is said $\frac{\alpha}{|\kappa|_k} = \beta \in \mathbb{D}^{+}$, then we have $\alpha = |\kappa|_k \cdot \beta$ and so

$$inf_{\mathbb{D}}\{|\kappa|_k \cdot \beta \in \mathbb{D}^{+}: \mathcal{D}_{u}(\beta) \preccurlyeq t\}$$

$$= inf_{\mathbb{D}}\{(|\kappa_1|e_1 + |\kappa_2|e_2) \cdot (\beta_1 e_1 + \beta_2 e_2) \in \mathbb{D}^+ : \mathcal{D}_u(\beta) \preccurlyeq t\}$$

$$= inf_{\mathbb{D}}\{(|\kappa_1|\beta_1)e_1 + (|\kappa_2|\beta_2)e_2 \in \mathbb{D}^+ : \mathcal{D}_u(\beta) \preccurlyeq t\}$$

$$= inf\{|\kappa_1|\beta_1 : D_{u_1}(\beta_1) \le t_1\}e_1 + inf\{|\kappa_2|\beta_2 : D_{u_2}(\beta_2) \le t_2\}e_2$$

$$= \left(|\kappa_1| inf\{\beta_1 : D_{u_1}(\beta_1) \le t_1\}\right)e_1 + \left(|\kappa_2| inf\{\beta_2 : D_{u_2}(\beta_2) \le t_2\}\right)e_2$$

$$= (|\kappa_1|e_1 + |\kappa_2|e_2)\left(inf\{\beta_1 : D_{u_1}(\beta_1) \le t_1\}e_1 + inf\{\beta_2 : D_{u_2}(\beta_2) \le t_2\}e_2\right)$$

$$= |\kappa|_k \cdot inf_{\mathbb{D}}\{\beta \in \mathbb{D}^+ : \mathcal{D}_u(\beta) \preccurlyeq t\}$$

$$= |\kappa|_k \cdot u^*(t).$$

d) If $|u|_k \prec |v|_k$, then the inequality $\mathcal{D}_u(\lambda) \preccurlyeq \mathcal{D}_v(\lambda)$ is obtained from the Theorem 1.

Since $\mathcal{D}_u(\lambda) \preccurlyeq \mathcal{D}_v(\lambda)$, we have

$$D_{u_1}(\lambda_1) \le D_{v_1}(\lambda_1) \wedge D_{u_2}(\lambda_2) \le D_{v_2}(\lambda_2).$$

This shows that

$$\{\lambda_1 \ge 0 : D_{v_1}(\lambda_1) \le t_1\} \subset \{\lambda_1 \ge 0 : D_{u_1}(\lambda_1) \le t_1\}$$

and

$$\{\lambda_2 \ge 0 : D_{v_2}(\lambda_2) \le t_2\} \subset \{\lambda_2 \ge 0 : D_{u_2}(\lambda_2) \le t_2\}$$

as a result

$$\{\lambda \in \mathbb{D}^+ : \mathcal{D}_v(\lambda) \preccurlyeq t\} \subset \{\lambda \in \mathbb{D}^+ : \mathcal{D}_u(\lambda) \preccurlyeq t\}$$

and so

$$inf_{\mathbb{D}}\{\lambda \in \mathbb{D}^+ : \mathcal{D}_u(\lambda) \preccurlyeq t\} \preccurlyeq inf_{\mathbb{D}}\{\lambda \in \mathbb{D}^+ : \mathcal{D}_v(\lambda) \preccurlyeq t\},$$

namely

$$u^*(t) \preccurlyeq v^*(t).$$

e) Let $p \in \mathbb{R}$ and $\lambda \in \mathbb{BC}$ then $(\lambda_1 e_1 + \lambda_2 e_2)^p = \lambda_1^p e_1 + \lambda_2^p e_2$ (Sağır & Değirmen, 2022).

Also in case of using the properties of $\mathcal{D}_u(\lambda)$

$$\mathcal{D}_{((|u|_k)^p)}(\lambda) = \mathcal{D}_{((|u_1|e_1+|u_2|e_2)^p)}(\lambda) = \mathcal{D}_{(|u_1|^p e_1+|u_2|^p e_2)}(\lambda)$$
$$= D_{|u_1|^p}(\lambda_1)e_1 + D_{|u_2|^p}(\lambda_2)e_2. \qquad (2.9)$$

Using the definitions of the distribution functions $D_{u_i}(\lambda_i)$ with $i = 1,2$ we have

$$D_{|u_i|^p}(\lambda_i) = \mu\{n \in \mathbb{N}: \lambda_i < ||u_i(n)|^p|\}$$
$$= \mu\left\{n \in \mathbb{N}: (\lambda_i)^{\frac{1}{p}} < |u_i(n)|\right\}$$
$$= D_{u_i}\left((\lambda_i)^{\frac{1}{p}}\right).$$

Therefore,

$$\mathcal{D}_{((|u|_k)^p)}(\lambda) = D_{u_1}\left((\lambda_1)^{\frac{1}{p}}\right)e_1 + D_{u_2}\left((\lambda_2)^{\frac{1}{p}}\right)e_2$$
$$= \mathcal{D}_u\left((\lambda)^{\frac{1}{p}}\right)$$

and hence

$$((|u|_k)^p)^*(t) = inf_{\mathbb{D}}\left\{\lambda \in \mathbb{D}^+: \mathcal{D}_{((|u|_k)^p)}(\lambda) \preccurlyeq t\right\}$$
$$= inf_{\mathbb{D}}\left\{\lambda \in \mathbb{D}^+: \mathcal{D}_u\left(\lambda^{\frac{1}{p}}\right) \preccurlyeq t\right\}$$
$$= inf_{\mathbb{D}}\{\alpha^p \in \mathbb{D}^+: \mathcal{D}_u(\alpha) \preccurlyeq t\}$$
$$= inf_{\mathbb{D}}\{(\alpha_1 e_1 + \alpha_2 e_2)^p \in \mathbb{D}^+: \mathcal{D}_u(\alpha) \preccurlyeq t\}$$
$$= inf_{\mathbb{D}}\{{\alpha_1}^p e_1 + {\alpha_2}^p e_2 \in \mathbb{D}^+: \mathcal{D}_u(\alpha_1 e_1 + \alpha_2 e_2) \preccurlyeq t\}.$$

Now if (2.5) is used,

$$((|u|_k)^p)^*(t) = inf\{{\alpha_1}^p: \alpha^p \in \mathbb{D}^+ \wedge \mathcal{D}_u(\alpha) \preccurlyeq t\}e_1$$
$$+ inf\{{\alpha_2}^p: \alpha^p \in \mathbb{D}^+ \wedge \mathcal{D}_u(\alpha) \preccurlyeq t\}e_2$$

$$= \left(inf\left\{\alpha_1 : \alpha^p \in \mathbb{D}^+ \ \wedge \ D_{u_1}(\alpha_1) \leq t_1\right\}\right)^p e_1 + \left(inf\left\{\alpha_2 : \alpha^p \in \mathbb{D}^+ \ \wedge \ D_{u_2}(\alpha_2) \leq t_2\right\}\right)^p e_2$$

$$= \left(\left(inf\left\{\alpha_1 : \alpha^p \in \mathbb{D}^+ \ \wedge \ D_{u_1}(\alpha_1) \leq t_1\right\}\right)e_1 + \left(inf\left\{\alpha_2 : \alpha^p \in \mathbb{D}^+ \ \wedge \ D_{u_2}(\alpha_2) \leq t_2\right\}\right)e_2\right)^p$$

$$= (inf_{\mathbb{D}}\{\alpha = \alpha_1 e_1 + \alpha_2 e_2 \in \mathbb{D}^+ : \mathcal{D}_u(\alpha_1 e_1 + \alpha_2 e_2) \preccurlyeq t)^p,$$

and so

$$((|u|_k)^p)^*(t) = (u^*)^p(t). \tag{2.10}$$

Definition 9. For a $\mathrm{H} \in \mathcal{G} = 2^{\mathbb{N}}$, the function defined by $\chi_{\mathrm{H}}(n) = \begin{cases} 1, n \in \mathrm{H} \\ 0, n \notin \mathrm{H} \end{cases}$, is the known characteristic function of the set H, therefore the function defined from $\mathbb{N}$ to $\mathbb{D}^+$ as

$$\chi_{\mathrm{H}}^{\mathbb{D}}(n) = \chi_{\mathrm{H}}(n)e_1 + \chi_{\mathrm{H}}(n)e_2$$

is called $\mathbb{D}$-characteristic function of the set H. At the time, it can be written by

$$\chi_{\mathrm{H}}^{\mathbb{D}}(n) = \begin{cases} 1e_1 + 1e_2, n \in \mathrm{H} \\ 0e_1 + 0e_2, n \notin \mathrm{H} \end{cases} = \begin{cases} 1, n \in \mathrm{H} \\ 0, n \notin \mathrm{H} \end{cases}.$$

Now let us construct the $\mathbb{D}$-distribution and $\mathbb{D}$-decreasing rearrangement functions of this function χ_{H}. Let $\lambda = (\lambda_1 e_1 + \lambda_2 e_2)$ be an element in $\mathbb{D}^+$, we have

$$\mathcal{D}_{\left(\chi_{\mathrm{H}}^{\mathbb{D}}(n)\right)}(\lambda) = \mathcal{D}_{(\chi_{\mathrm{H}}(n)e_1 + \chi_{\mathrm{H}}(n)e_2)}(\lambda_1 e_1 + \lambda_2 e_2)$$
$$= D_{\chi_{\mathrm{H}}(n)}(\lambda_1)e_1 + D_{\chi_{\mathrm{H}}(n)}(\lambda_2)e_2.$$

Using the definitions of the distribution functions $D_{u_i}(\lambda_i)$ with $i = 1,2$ we have

$$D_{\chi_{\mathrm{H}}(n)}(\lambda_i) = \begin{cases} \mu(H) & ,0 \leq \lambda_i < 1 \\ 0 & ,\lambda_i \geq 1 \end{cases}.$$

Finally,

$$\mathcal{D}_{\left(\chi_{H}^{\mathbb{D}}(n)\right)}(\lambda)=\begin{cases}\mu(H)e_1+\mu(H)e_2 & ,0\le\lambda_1<1\wedge\ 0\le\lambda_2<1\\ \mu(H)e_1+0e_2 & ,0\le\lambda_1<1\wedge\ \lambda_2\ge 1\\ 0e_1+\mu(H)e_2 & ,\lambda_1\ge 1\wedge\ 0\le\lambda_2<1\\ 0e_1+0e_2 & ,\lambda_1\ge 1\wedge\ \lambda_2\ge 1\end{cases}$$

$$\mathcal{D}_{\left(\chi_{H}^{\mathbb{D}}(n)\right)}(\lambda)=\begin{cases}\mu(H) & ,\quad 0\le\lambda_1<1\wedge\ 0\le\lambda_2<1\\ \mu(H)e_1 & ,\quad 0\le\lambda_1<1\wedge\ \lambda_2\ge 1\\ \mu(H)e_2 & ,\quad \lambda_1\ge 1\wedge\ 0\le\lambda_2<1\\ 0 & ,\quad \lambda_1\ge 1\wedge\ \lambda_2\ge 1\end{cases}\tag{2.11}$$

is obtained.

Now, if the $\mathbb{D}$-decreasing rearrangement function of the $\mathbb{D}$-characteristic function $\chi_{H}^{\mathbb{D}}$ is written according to (2.11), it is clearly seen that

$$\begin{aligned}\left(\chi_{H}^{\mathbb{D}}\right)^{*}(t)&=inf_{\mathbb{D}}\left\{\lambda\in\mathbb{D}^{+}:\mathcal{D}_{\left(\chi_{H}^{\mathbb{D}}(n)\right)}(\lambda)\preccurlyeq t\right\}\\ &=inf_{\mathbb{D}}\left\{\lambda_1e_1+\lambda_2e_2\in\mathbb{D}^{+}:\mathcal{D}_{\left(\chi_{H}^{\mathbb{D}}(n)\right)}(\lambda)\preccurlyeq t_1e_1+t_2e_2\right\}\\ &=inf\left\{\lambda_1\ge 0:\ D_{\chi_{H}(n)}(\lambda_1)\le t_1\right\}e_1\\ &\qquad+inf\left\{\lambda_2\ge 0:\ D_{\chi_{H}(n)}(\lambda_2)\le t_2\right\}e_2\\ &=\left(\chi_{H}\right)^{*}(t_1)e_1+\left(\chi_{H}\right)^{*}(t_2)e_2\end{aligned}$$

and so

$$\left(\chi_{H}^{\mathbb{D}}\right)^{*}(t_1e_1+t_2e_2)=\begin{cases}1e_1+1e_2 & ,\quad t_1<\mu(H)\wedge\ t_2<\mu(H)\\ 1e_1+0e_2 & ,\quad t_1<\mu(H)\wedge\ t_2\ge\mu(H)\\ 0e_1+1e_2 & ,\quad t_1\ge\mu(H)\wedge\ t_2<\mu(H)\\ 0e_1+0e_2 & ,\quad t_1\ge\mu(H)\wedge\ t_2\ge\mu(H)\end{cases}$$

$$= \begin{cases} 1 \ , \ t_1 < \mu(H) \wedge t_2 < \mu(H) \\ e_1 \ , \ t_1 < \mu(H) \wedge t_2 \geq \mu(H) \\ e_2 \ , \ t_1 \geq \mu(H) \wedge t_2 < \mu(H) \\ 0 \ , \ t_1 \geq \mu(H) \wedge t_2 \geq \mu(H) \end{cases},$$

also

$$\left(\chi_{\mathrm{H}}^{\mathbb{D}}\right)^*(t) = \chi_{[1,\mu(\mathrm{H}))}(t_1)e_1 + \chi_{[1,\mu(\mathrm{H}))}(t_2)e_2. \qquad (2.12)$$

Theorem 3. Let $H \in \mathcal{G} = 2^{\mathbb{N}}$ and $u = (u(n))$ is a sequence in $\mathcal{W}_{\mathbb{BC}}$. For $t \in \mathbb{D}^+$,

$$\left(u \cdot \chi_H\right)^*(t) \preccurlyeq u^*(t).$$

Proof. Since $\left|u \cdot \chi_H\right|_k \preccurlyeq |u|_k$, by Theorem 1.(d), we have

$$\left(u \cdot \chi_H\right)^*(t) \preccurlyeq u^*(t).$$

Lemma 2. Let $(G, \mathcal{G}, \mu)$ is be a measurement space further g and h are be two measurable functions. Then the inequalities

$$(g + h)^*(\alpha + \beta) \leq g^*(\alpha) + h^*(\beta)$$

and

$$(g \cdot h)^*(\alpha + \beta) \leq g^*(\alpha) \cdot h^*(\beta)$$

hold for all $\alpha, \beta \geq 0$ (Castillo and Rafeiro, 2015).

Theorem 4. Let $(\mathbb{N}, \mathcal{G}, \mu)$ be a measure space and $u = (u(n))$, $v = (v(n))$ be two sequence in $\mathcal{W}_{\mathbb{BC}}$. Then, the $\mathbb{D}$-inequalities

$$(u + v)^*(\lambda + \delta) \preccurlyeq u^*(\lambda) + v^*(\delta) \qquad (2.13)$$

and

$$(u \cdot v)^*(\lambda + \delta) \preccurlyeq u^*(\lambda) \cdot v^*(\delta) \qquad (2.14)$$

hold for all $\lambda, \delta \in \mathbb{D}^+$ with $\lambda = \lambda_1 e_1 + \lambda_2 e_2$ and $\delta = \delta_1 e_1 + \delta_2 e_2$.

Proof. Since $u^*(\lambda) = u_1^*(\lambda_1)e_1 + u_2^*(\lambda_2)e_2$ and

$v^*(\delta) = v_1^*(\delta_1)e_1 + v_2^*(\delta_2)e_2$ from (2.8) and also if use Lemma 2, we have

$$(u+v)^*(\lambda+\delta) = \big((u_1e_1+u_2e_2)+(v_1e_1+v_2e_2)\big)^*(\lambda+\delta)$$
$$= \big((u_1+v_1)e_1+(u_2+v_2)e_2\big)^*\big((\lambda_1+\delta_1)e_1+(\lambda_2+\delta_2)e_2\big)$$
$$= (u_1+v_1)^*(\lambda_1+\delta_1)e_1+(u_2+v_2)^*(\lambda_2+\delta_2)e_2$$
$$\preccurlyeq \big(u_1^*(\lambda_1)+v_1^*(\delta_1)\big)e_1+\big(u_2^*(\lambda_2)+v_2^*(\delta_2)\big)e_2$$
$$= (u_1^*(\lambda_1)e_1+u_2^*(\lambda_2)e_2)+\big(\ v_1^*(\delta_1)e_1+v_2^*(\delta_2)\big)e_2$$
$$= u^*(\lambda)+v^*(\delta).$$

Again, we have

$$(u\cdot v)^*(\lambda+\delta) = \big((u_1e_1+u_2e_2)\cdot(v_1e_1+v_2e_2)\big)^*(\lambda+\delta)$$
$$= \big((u_1\cdot v_1)e_1+(u_2\cdot v_2)e_2\big)^*\big((\lambda_1+\delta_1)e_1+(\lambda_2+\delta_2)e_2\big)$$
$$= (u_1\cdot v_1)^*(\lambda_1+\delta_1)e_1+(u_2\cdot v_2)^*(\lambda_2+\delta_2)e_2$$
$$\preccurlyeq \big(u_1^*(\lambda_1)\cdot v_1^*(\delta_1)\big)e_1+\big(u_2^*(\lambda_2)\cdot v_2^*(\delta_2)\big)e_2$$
$$= (u_1^*(\lambda_1)e_1+u_2^*(\lambda_2)e_2)\cdot\big(\ v_1^*(\delta_1)e_1+v_2^*(\delta_2)\big)e_2$$
$$= u^*(\lambda)\cdot v^*(\delta).$$

If $\lambda = \delta = \frac{\alpha}{2}$ is chosen in (2.13) and (2.14), then

$$(u+v)^*(\alpha) \preccurlyeq u^*\left(\frac{\alpha}{2}\right)+v^*\left(\frac{\alpha}{2}\right)$$

and

$$(u\cdot v)^*(\alpha) \preccurlyeq u^*\left(\frac{\alpha}{2}\right)\cdot v^*\left(\frac{\alpha}{2}\right)$$

is obtained.

Definition 10. $u = (u(n))$ be a sequence in $\mathcal{W}_{\mathbb{BC}}$. The function defined by

$$u^{**}(n)=\frac{1}{n}\sum_{x=1}^{n}u^{*}(x)$$

is called the $\mathbb{D}$-average function of u^{*}.

Theorem 5. Let $u=(u(n))$ be a sequence in $\mathcal{W}_{\mathbb{BC}}$. The $\mathbb{D}$-average function of u^{*} has the following features:

a) For every $n=1,2,3\ldots$, $u^{*}(n)\preccurlyeq u^{**}(n)$.

b) u^{**} is $\mathbb{D}$-decreasing.

c) If n is an even natural number,

$$(u+v)^{**}(n)\preccurlyeq u^{**}(n/2)+v^{**}(n/2).$$

d) If n is an odd natural number and $[\![.]\!]$ is an greatest integer function, then

$$(u+v)^{**}(n)\preccurlyeq\frac{n+1}{n}\left(u^{**}([\![n/2]\!]+1)+v^{**}([\![n/2]\!]+1)\right).$$

Proof. a) Since u^{*} a $\mathbb{D}$-decreasing function and $n=ne_1+ne_2$ for all $n\geq 1$,

$$u^{**}(n)=\frac{1}{n}\sum_{x=1}^{n}u^{*}(x)=\frac{1}{n}\Big(\big(u^{*}(1)+u^{*}(2)+\cdots+u^{*}(n)\big)\Big)$$
$$\succcurlyeq\frac{1}{n}\big(n\cdot u^{*}(n)\big)=u^{*}(n)$$

is obtained.

b) For $n\geq 1$, we have

$$u^{**}(n+1)=\frac{1}{n+1}\sum_{x=1}^{n+1}u^{*}(x)$$
$$=\frac{1}{n+1}\left(\sum_{x=1}^{n}u^{*}(x)+u^{*}(n+1)\right)$$

$$= \frac{1}{n+1}\sum_{x=1}^{n} u^*(x) + \frac{1}{n+1} u^*(n+1). \quad (2.15)$$

Since $u^*(n+1) \preccurlyeq u^*(i)$ for $1 \leq i \leq n$, there exist

$$u^*(n+1) \preccurlyeq \frac{1}{n}\big(u^*(1) + u^*(2) + \cdots + u^*(n)\big).$$

If this inequality is written in (2.15), we have

$$u^{**}(n+1) \preccurlyeq \frac{1}{n+1}\sum_{x=1}^{n} u^*(x) + \frac{1}{n(n+1)}\big(u^*(1) + \cdots + u^*(n)\big) = u^{**}(n).$$

As a result, u^{**} is $\mathbb{D}$-decreasing.

c) If the $\mathbb{D}$-average function definition and (2.13) are used,

$$\begin{aligned}(u+v)^{**}(n) &= \frac{1}{n}\sum_{x=1}^{n}(u+v)^*(x) \\ &\preccurlyeq \frac{1}{n}\sum_{x=1}^{n}\left(u^*\left(\frac{x}{2}\right) + v^*\left(\frac{x}{2}\right)\right) \\ &= \frac{1}{n}\left\{\sum_{x=1}^{n} u^*\left(\frac{x}{2}\right) + \sum_{x=1}^{n} v^*\left(\frac{x}{2}\right)\right\}.\end{aligned}$$

Since $(n-1) \leq t_i \prec n$, we have $u_i^*(t_i) = u_i^*(n)$ with $i = 1, 2$ (Castillo & Rafeiro, 2015).

For $(n-1) \preccurlyeq t \prec n$, we have

$$u^*(t) = u_1^*(t_1)e_1 + u_2^*(t_2)e_2 = u_1^*(n)e_1 + u_2^*(n)e_2 = u^*(n).$$

Therefore,

$$\sum_{x=1}^{n} u^*\left(\frac{x}{2}\right)$$

$$= u^*\left(\frac{1}{2}\right) + u^*\left(\frac{2}{2}\right) + u^*\left(\frac{3}{2}\right) + u^*\left(\frac{4}{2}\right) + \cdots + u^*\left(\frac{n}{2}\right)$$

$$= u^*(1) + u^*(1) + u^*(2) + u^*(2) + \cdots + u^*\left(\frac{n}{2}\right)$$

$$= 2\left\{u^*(1) + u^*(2) + \cdots + u^*\left(\frac{n}{2}\right)\right\}$$

$$= 2\sum_{x=1}^{n/2} u^*(x)$$

similarly,

$$\sum_{x=1}^{n} v^*\left(\frac{x}{2}\right) = 2\sum_{x=1}^{n/2} v^*(x).$$

As a result,

$$(u+v)^{**}(n) \leqslant \frac{1}{n}\left\{2\sum_{x=1}^{n/2} u^*(x) + 2\sum_{x=1}^{n/2} v^*(x)\right\}$$

$$= \frac{1}{n/2}\left\{\sum_{x=1}^{n/2} u^*(x) + \sum_{x=1}^{n/2} v^*(x)\right\}$$

$$= u^{**}(n/2) + v^{**}(n/2).$$

d) Assume that n is the odd natural number. When using the proof method in c,

$$(u+v)^{**}(n) = \frac{1}{n}\sum_{x=1}^{n} (u+v)^*(x)$$

$$\leqslant \frac{1}{n}\sum_{x=1}^{n}\left(u^*\left(\frac{x}{2}\right) + v^*\left(\frac{x}{2}\right)\right)$$

$$
\begin{aligned}
&= \frac{1}{n}\sum_{x=1}^{n}\left(u^*\left(\frac{x}{2}\right)\right) + \frac{1}{n}\sum_{x=1}^{n}\left(v^*\left(\frac{x}{2}\right)\right) \\
&= \frac{1}{n}\left\{u^*\left(\frac{1}{2}\right) + u^*\left(\frac{2}{2}\right) + u^*\left(\frac{3}{2}\right) + \cdots + u^*\left(\frac{n-1}{2}\right) + u^*\left(\frac{n}{2}\right)\right\} \\
&\quad + \frac{1}{n}\left\{v^*\left(\frac{1}{2}\right) + v^*\left(\frac{2}{2}\right) + v^*\left(\frac{3}{2}\right) + \cdots + v^*\left(\frac{n-1}{2}\right) + v^*\left(\frac{n}{2}\right)\right\} \\
&\preccurlyeq \frac{1}{n}\left\{u^*\left(\frac{1}{2}\right) + u^*\left(\frac{2}{2}\right) + \cdots + u^*\left(\frac{n-1}{2}\right) + u^*\left(\frac{n}{2}\right) + u^*\left(\frac{n}{2}\right)\right\} \\
&\quad + \frac{1}{n}\left\{v^*\left(\frac{1}{2}\right) + v^*\left(\frac{2}{2}\right) + \cdots + v^*\left(\frac{n-1}{2}\right) + v^*\left(\frac{n}{2}\right) + v^*\left(\frac{n}{2}\right)\right\} \\
&= \frac{2}{n}\sum_{x=1}^{[\![n/2]\!]+1} u^*(x) + \frac{2}{n}\sum_{x=1}^{[\![n/2]\!]+1} v^*(x) \\
&= \frac{2}{n}\frac{\{[\![n/2]\!]+1\}}{\{[\![n/2]\!]+1\}}\sum_{x=1}^{[\![n/2]\!]+1}\left(u^*(x) + v^*(x)\right) \\
&= \frac{2\left\{\left[\!\left[\frac{n+1}{2} - \frac{1}{2}\right]\!\right] + 1\right\}}{n}\left\{\frac{1}{([\![n/2]\!]+1)}\sum_{k=1}^{[\![n/2]\!]+1} u^*(x)\right. \\
&\qquad\qquad \left. + \frac{1}{([\![n/2]\!]+1)}\sum_{x=1}^{[\![n/2]\!]+1} v^*(x)\right\} \\
&= \frac{2\left\{\left(\frac{n+1}{2} - 1\right) + 1\right\}}{n}\left(u^{**}([\![n/2]\!]+1) + v^{**}([\![n/2]\!]+1)\right) \\
&= \frac{n+1}{n}\left(u^{**}([\![n/2]\!]+1) + v^{**}([\![n/2]\!]+1)\right)
\end{aligned}
$$

is obtained.

References

[1] Arora, S.C., Datt, G. & Verma, S. (2009). Operators on Lorentz sequence spaces. *Math. Bohem.*,*No:*1, 87-98.

[2] Bennett, C. & Sharpley, R. (1988). *Interpolation operators*. Toronto: Academic Press Inc.

[3] Castillo, R.E. & Rafeiro, H. (2016). *An Introductory course in Lebesque spaces*. Switzerland: Springer.

[4] Değirmen, N. & Sağır, B. (2022). *On bicomplex* $\mathbb{BC}-modules$ $l_p^k(\mathbb{BC})$ *and some of their geometric properties*.Georgia: Georgian Mathematical Journal.

[5] Emel'yanov, E. (2007). *Introductıon to measure theory and Lebesgue ıntegration*. Ankara: Mıddle East Technical University Press.

[6] Eryılmaz, İ. & Işık, G. (2019). Multiplication Operators On Grand Lorentz Spaces .Kayseri: *Erciyes University Institute of Science and Technology Journal of Science* , 35 (1) , 41-49 .

[7] Halmos, P.R. (1978). *Measure Theory*. New York: Springer-Verlag.

[8] Hamilton, W.R. (1844). On quaternions, or on a new system of imaginaries in algebra. *Philosophical Magazine*. Vol. 25, no. 3, 489–495.

[9] Kreysig, E. (1978) *Introductory Functional Analysis with Applications*. New York: John Wiley and Sons.

[10] Lorentz, G. G. (1950). Some new functional spaces, *Ann. of Math*. 2(51), 37-55.

[11] Lorentz, G. G. (1951). On the theory of spaces $\wedge$, *Pacific J. Math*. 1, 411-429.

[12] Luna-Elizarrarás, M.E. & et al. (2015). *Bicomplex holomorphic functions: The algebra, geometry and analysis of*

bicomplex numbers. Switzerland: Springer International Publishing DOI 10.1007/978-3-319-24868-4-2.

[13] Oğur, O. & Duyar, C. (2016). On generalized Lorentz sequence space defined by modulus functions. *Filomat*, 30(2), 497-504.

[14] Price, G.B. (1991). *An Introduction to Multicomplex Spaces and Functions. Monographs and Textbooks in Pure and Applied Mathematics*, 140, New York: Marcel Dekker, Inc.

[15] Riley, J.D. (1953). Contributions to the theory of functions of bicomplex variable. *Tohoku Math. J*. v. 2, 132–165.

[16] Rudin, W. (1953). *Principles of Mathematical Analysis*. (3th ed.). New York: Mc Graw-Hill, Inc.

[17] Scorza Dragoni, G. (1934). Sulle funzioni olomorfe di una variabile bicomplessa. Reale Accad. d'Italia, *Mem. Classe Sci. Nat. Fis. Mat*. v. 5, 597–665.

[18] Spampinato, N. (1935). Estensione nel campo bicomplesso di due teoremi, del Levi-Civita e del Severi, per le funzioni olomorfe di due variabili bicomplesse I, II. *Reale Accad. Naz. Lincei*, v. 22 No. 38–43, 96–102.

[19] Spampinato, N. (1936). Sulla rappresentazione di funzioni di variabile bicomplessa totalmente derivabili. *Ann. Mat. Pura Appli*. v. 14, 305–325.

CHAPTER II

Q Tensor on Para-Sasakian Manifolds

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Introduction

One of the most comprehensive and active theories of modern differential geometry is Riemannian manifolds. Therefore, such manifolds have been examined and studied frequently by several mathematicians in literature in recent years. Over time, different structures have been defined on Riemannian manifolds and thanks

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to these structures, some special different manifolds have been introduced to the literature. One of these manifolds is the almost paracontact manifolds and their subclasses defined by Sato (Sato, 1976). Since then, many authors have made significant contributions to these manifolds.

As a special subclass of almost paracontact manifolds, para-Sasakian manifolds represent a captivating and significant area of study in the realm of differential geometry and mathematics. Such manifolds are a natural extension of Sasakian manifolds, and they possess remarkable geometric properties that have far-reaching implications across various mathematical disciplines.

Para-Sasakian manifolds are closely related to the broader family of Sasakian manifolds. A Sasakian manifold is a Riemannian manifold equipped with a special contact structure known as the Reeb foliation. What distinguishes para-Sasakian manifolds is their compatibility with a pseudo-Riemannian metric alongside the Reeb foliation, resulting in a perfect blend of Riemannian and Lorentzian geometries. This combination of geometric structures provides a unique setting for the exploration of mathematical concepts.

On the other hand, using the Riemannian metric on manifolds plays an important role in examining and characterizing manifolds and their submanifolds. While examining Riemannian manifolds and their submanifolds, curvatures are generally used, classifications are made with the help of these curvatures, and characterization theorems and results are given depending on the curvatures. While investigating Riemannian manifolds and their submanifolds, one of the new methods is to make use of certain curvature tensor conditions on the manifolds. There are many interesting paper about various type of manifolds satisfying curvature conditions.

In their work (Adati & Matsumoto, 1977), T. Adati and K. Matsumoto defined the concepts of para-Sasakian and special para-Sasakian manifolds, which are regarded as specific instances of almost paracontact manifolds initially established by I. Sato (Sato, 1976). Within the same study, the authors conducted an investigation into conformally symmetric para-Sasakian manifolds, establishing the result that a para-Sasakian manifold of dimension n (where $n > 3$) that is conformally symmetric is both conformally flat and special para-Sasakian. Subsequently, in their study (De & Guha, 1992), the authors examined para-Sasakian manifolds satisfying the condition Weyl-semisymmetric, and their findings demonstrated that a Weyl-semisymmetric manifold of dimension n that is also para-Sasakian exhibits conformal flatness. This research is devoted to achieving classifications of para-Sasakian manifolds with Weyl-pseudosymmetric condition, which represent an expanded class of Weyl-semisymmetric manifolds that have para-Sasakian structure, as well as additional characterizations of para-Sasakian manifolds that satisfy the condition curvature $C \cdot S = 0$.

In his research, Özgür (Özgür, 2005) explored Weyl-pseudosymmetric manifolds that have para-Sasakian structure. Also, in same research he investigate para-Sasakian manifolds under the curvature condition $C \cdot S = 0$. Para-Sasakian manifolds have been the subject of study by various authors, including Deshmukh and Ahmed (Desmukh & Ahmed, 1980), De et al. (De & et al., 2008), Matsumoto, Ianus, and Mihai (Matsumoto, Ianus & Mihai, 1986), Sharfuddin, Deshmukh, and Husain (Sharfuddin, Deshmukh & Husain, 1980), De et al. (De, Han & Mandal, 2017), Adati and Miyazawa (Adati & Miyazawa, 1979), Acet et al. (Acet, Kılıç & Yüksel Perktaş, 2012), Ozgür and Tripathi (Özgür & Tripathi, 2007) among several others.

Motivated and inspired by the above studies, in this work we deal with Q curvature tensor on para-Sasakian manifolds, which is a fruitful topic of paracontact manifolds, and we obtain some significant characterization theorems depending on this tensor. Our work is structured as follows. First section is dedicated to the introduction, which contains a background of manifolds. In the next section, in preliminaries, we give some requirement notions and formulas that we will make use of in the proof of our main results. The final section, as main results, we give some characteriztions on para-Sasakian manifolds which are equipped with the curvature tensor Q.

Preliminaries

In this section, we collect some necessary notions and formulas related to almost paracontact metric manifolds and para-Sasakian manifolds which will be used later in the next section.

Let M be a contact manifold of dimension $2n+1$ together with a contact form η such that $\eta \wedge (d\eta)^n \neq 0$. It is widely recognized that a contact manifold admits a special vector field ξ , which is said to be characteristic vector field of the contact manifold, satisfying $\eta(\xi)=1$ and $d\eta(\xi,\daleth_1)=0$ for all $\daleth_1$ in $\Gamma(TM)$. Furthermore, M possesses a Riemannian metric g and a $(1,1)-$type tensor field φ satisfying $\varphi^2 = I-\eta\otimes\xi$, $g(\daleth_1,\xi)=\eta(\daleth_1)$, and $g(\daleth_1,\varphi\daleth_2)=d\eta(\daleth_1,\daleth_2)$. In this context, we refer to (φ,ξ,η,g) as a contact metric structure. A manifold which is endowed with a such structure is termed Sasakian if

$$\left(\nabla_{\daleth_1}\varphi\right)\daleth_2 = g(\daleth_1,\daleth_2)\xi-\eta(\daleth_2)\daleth_1,$$

in which case

$$\nabla_{\daleth_1}\xi = -\varphi\daleth_1$$

and

$$R(\daleth_1,\daleth_2)\xi = \eta(\daleth_2)\daleth_1-\eta(\daleth_1)\daleth_2$$

for all vector fields $\daleth_1,\daleth_2 \epsilon \Gamma(TM)$. Here, ∇ denotes the connection of the manifold, which is called the Levi-Civita connection.

An almost paracontact manifold M having dimension $2n+1$ is a differentiable manifold consisting of the structure (φ,ξ,η,g), where φ represents a $(1,1)-$tensor field, ξ is a vector field, η is a $1-$form and g is the Riemannian metric on M. This structure is characterized by the following conditions (Sato, 1976)

$$\varphi^2\daleth_1 = \daleth_1 - \eta(\daleth_1)\xi, \qquad \ldots(1)$$

$$\eta(\xi) = 1, \qquad \ldots(2)$$

$$\varphi\xi = 0, \qquad \ldots(3)$$

$$\eta\varphi = 0, \qquad \ldots(4)$$

and

$$g(\xi, \daleth_1) = \eta(\daleth_1), \qquad \ldots(5)$$

$$g(\varphi\daleth_1, \varphi\daleth_2) = g(\daleth_1, \daleth_2) - \eta(\daleth_1)\eta(\daleth_2) \qquad \ldots(6)$$

for all vector fields $\daleth_1, \daleth_2 \epsilon \Gamma(TM)$, where $\Gamma(TM)$ denotes the set of all vector field on the manifold M.

It is to be noted that as a direct consequence of the equations (1), (3), (4) and (6) we also have

$$g(\varphi\daleth_1, \daleth_2) + g(\daleth_1, \varphi\daleth_2) = 0. \qquad \ldots(7)$$

If the structure (φ, ξ, η, g) on the manifold M satisfies

$$d\eta = 0, \qquad \ldots(8)$$

$$\nabla_{\daleth_1}\xi = \varphi\daleth_1, \qquad \ldots(9)$$

$$(\nabla_{\daleth_1}\varphi)\daleth_2 = -g(\daleth_1, \daleth_2)\xi - \eta(\daleth_2)\daleth_1 + 2\eta(\daleth_1)\eta(\daleth_2)\xi, \qquad \ldots(10)$$

then M is said to define a para-Sasakian manifold, or shortly, a *P*-Sasakian manifold (Adati & Matsumuto, 1977). We denote a para-Sasakian manifold by $(M, \varphi, \xi, \eta, g)$.

Also we want to remark that in a para-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ the following properties are satisfied (Zamkovoy, 2009):

$$Ric\ \xi = -2n. \qquad \ldots(11)$$

$$S(\daleth_1, \xi) = -2n\eta(\daleth_1), \qquad \ldots(12)$$

$$R(\daleth_1, \daleth_2)\xi = \eta(\daleth_1)\daleth_2 - \eta(\daleth_2)\daleth_1, \qquad \ldots(13)$$

$$R(\xi, \daleth_1)\daleth_2 = \eta(\daleth_2)\daleth_1 - g(\daleth_1, \daleth_2)\xi \qquad \ldots(14)$$

$$R(\xi, \daleth_1)\xi = \daleth_1 - \eta(\daleth_1)\xi, \qquad \ldots(15)$$

$$g\,(R(\daleth_1,\daleth_2)\daleth_3,\xi) = \eta(R(\daleth_1,\daleth_2)\daleth_3) = g\,(\daleth_1,\daleth_3)\eta(\daleth_2) - g\,(\daleth_2,\daleth_3)\eta(\daleth_1) \qquad \ldots \qquad (16)$$

for all vector fields $\daleth_1,\daleth_2,\daleth_3 \epsilon \Gamma(TM)$. Here, R is the Riemannian curvature tensor of the manifold, which is given by the formula

$$R(\daleth_1,\daleth_2)\daleth_3 = \nabla_{\daleth_1}\nabla_{\daleth_2}\daleth_3 - \nabla_{\daleth_2}\nabla_{\daleth_1}\daleth_3 - \nabla_{[\daleth_1,\daleth_2]}\daleth_3$$

and S stands for the Ricci tensor defined by $S(\daleth_1,\daleth_2) = g(Ric\daleth_1,\daleth_2)$, where Ric is the Ricci operator.

Additionally if the Ricci tensor S of a para-Sasakian manifold (M,φ,ξ,η,g) satisfies

$$S(\daleth_1,\daleth_2) = \rho g(\daleth_1,\daleth_2) + \delta\eta(\daleth_1)\eta(\daleth_2)$$

then the manifold is named as $\eta-$Einstein, where ρ and δ are smooth functions on the manifold M. An $\eta-$Einstein manifold becomes Einstein if $\delta=0$ (Adati & Miyazawa, 1979).

On the other hand Mantica and Suh defined a new curvature tensor, which is named as Q curvature tensor, given by (Mantica & Suh, 2013)

$$Q(\daleth_1,\daleth_2)\daleth_3 = R(\daleth_1,\daleth_2)\daleth_3 - \frac{\psi}{2n}[g(\daleth_2,\daleth_3)\daleth_1 - g(\daleth_1,\daleth_3)\daleth_2]. \qquad (17)$$

Here, ψ denotes an arbitrary scalar function. When $\psi = \frac{r}{(2n+1)}$, then this curvature tensor reduces to concircular curvature tensor.

Also it follows from the equations (13), (14), (15) and (17) one immediately has

$$Q(\daleth_1,\daleth_2)\xi = \left(1+\frac{\psi}{2n}\right)R(\daleth_1,\daleth_2)\xi \qquad \ldots(18)$$

$$Q(\xi,\daleth_2)\daleth_3 = \left(1+\frac{\psi}{2n}\right)R(\xi,\daleth_2)\daleth_3 \qquad \ldots(19)$$

$$Q(\xi,\daleth_2)\xi = \left(1+\frac{\psi}{2n}\right)R(\xi,\daleth_2)\xi \qquad \ldots(20)$$

for any vector fields $\daleth_1$ and $\daleth_2$ tangent to M. The curvature tensor Q has been studied by various researchers, including Yadav and Yıldız

(Yadav & Yıldız, 2022), De and Majhi (De and Majhi, 2019), Yıldırım (Yıldırım, 2022), Yılmaz (Bağdatli Yilmaz, 2020) as well as numerous other scholars.

Main Results

In this section we give our main results that are obtained in this work. We obtain that under some curvature conditions a para-Sasakian manifold (M,φ,ξ,η,g) becomes a constant negative curvature manifold.

The first result of this section is in the following.

Theorem 1: Let (M,φ,ξ,η,g) be a para-Sasakian manifold admitting curvature condition $Q(\xi,\daleth_4).Q=0$. Then, we have that either $\psi=-2n$ or the manifold M has constant negative curvature -1.

Proof: Let us suppose that the structure (φ,ξ,η,g) of the manifold M satisfies the condition $Q(\xi,\daleth_4).Q=0$, that is

$$(Q(\xi,\daleth_4)Q).(\daleth_1,\daleth_2)\daleth_5=0$$

for all vector fields $\daleth_1,\daleth_2,\daleth_4,\daleth_5\epsilon\Gamma(TM)$. This means that

$$Q(\xi,\daleth_4).Q(\daleth_1,\daleth_2)\daleth_5-Q(Q(\xi,\daleth_4)\daleth_1,\daleth_2)\daleth_5$$
$$Q(\daleth_1,Q(\xi,\daleth_4)\daleth_2)\daleth_5-Q(\daleth_1,\daleth_2)Q(\xi,\daleth_4)\daleth_5=0. \qquad (21)$$

Setting $\daleth_5=\xi$ in (21), one can write

$$Q(\xi,\daleth_4)Q(\daleth_1,\daleth_2)\xi-Q(Q(\xi,\daleth_4)\daleth_1,\daleth_2)\xi$$
$$Q(\daleth_1,Q(\xi,\daleth_4)\daleth_2)\xi-Q(\daleth_1,\daleth_2)Q(\xi,\daleth_4)\xi=0. \qquad \ldots (22)$$

For the first term of (22), using (18) and (19), we get

$$Q(\xi,\daleth_4)Q(\daleth_1,\daleth_2)\xi=\left(1+\frac{\psi}{2n}\right)^2 R(\xi,\daleth_4)R(\daleth_1,\daleth_2)\xi \qquad (23)$$

Also, making use of (19) in (23) we find that

$$Q(\xi,\daleth_4)Q(\daleth_1,\daleth_2)\xi = \left(1+\frac{\psi}{2n}\right)^2 (\eta(\daleth_2)g(\daleth_4,\daleth_1)\xi - \eta(\daleth_1)g(\daleth_4,\daleth_2)\xi). \tag{24}$$

For the second term of (22), by means of (19) we achieve

$$Q(Q(\xi,\daleth_4)\daleth_1,\daleth_2)\xi = \left(1+\frac{\psi}{2n}\right)^2 R(R(\xi,\daleth_4)\daleth_1,\daleth_2)\xi,$$

which together with the equations (13) and (15) takes the form

$$Q(Q(\xi,\daleth_4)\daleth_1,\daleth_2)\xi = \left(1+\frac{\psi}{2n}\right)^2 \begin{bmatrix} \eta(\daleth_1)\eta(\daleth_4)\daleth_2 - \eta(\daleth_1)\eta(\daleth_2)\daleth_4 \\ +g(\daleth_4,\daleth_1)\eta(\daleth_2)\xi - g(\daleth_4,\daleth_1)\daleth_2 \end{bmatrix} \tag{25}$$

For the third term of the equation (22), with the help of (18) and (19), we arrive at

$$Q(Q(\xi,\daleth_4)\daleth_1,\daleth_2)\xi = \left(1+\frac{\psi}{2n}\right)^2 R(\daleth_1,R(\xi,\daleth_4)\daleth_2)\xi. \qquad \ldots(26)$$

It follow from (13) and (15), the equation (25) becomes

$$Q(Q(\xi,\daleth_4)\daleth_1,\daleth_2)\xi = \left(1+\frac{\psi}{2n}\right)^2 \begin{bmatrix} \eta(\daleth_2)\eta(\daleth_1)\daleth_4 - \eta(\daleth_2)\eta(\daleth_4)\daleth_1 \\ +g(\daleth_4,\daleth_2)\daleth_1 - g(\daleth_4,\daleth_2)\eta(\daleth_1)\xi \end{bmatrix}. \tag{27}$$

For the fourth term of the equation (22), using (20) we write

$$Q(\daleth_1,\daleth_2)Q(\xi,\daleth_4)\xi = \left(1+\frac{\psi}{2n}\right) Q(\daleth_1,\daleth_2)R(\xi,\daleth_4)\xi \qquad \ldots(28)$$

In wiew of (15), we provide

$$Q(\daleth_1,\daleth_2)Q(\xi,\daleth_4)\xi = \left(1+\frac{\psi}{2n}\right) Q(\daleth_1,\daleth_2)(\daleth_4 - \eta(\daleth_4)\xi) \qquad \ldots (29)$$

Using (18) in (29), we obtain that

$$Q(\daleth_1,\daleth_2)Q(\xi,\daleth_4)\xi = \left(1+\frac{\psi}{2n}\right) Q(\daleth_1,\daleth_2)\daleth_4$$

$$\left(1+\frac{\psi}{2n}\right)^2 [\eta(\daleth_4)\eta(\daleth_1)\daleth_2 - \eta(\daleth_4)\eta(\daleth_2)\daleth_1]. \qquad \ldots(30)$$

Now, substituting the equations (24), (25), (27) and (30) into (22), we have

$$\left(1+\frac{\psi}{2n}\right)[R(\daleth_1,\daleth_2)\daleth_4+g(\daleth_2,\daleth_4)\daleth_1-g(\daleth_1,\daleth_4)\daleth_2]=0,$$

which implies that either

$$\frac{\psi}{2n}+1=0$$

or

$$R(\daleth_1,\daleth_2)\daleth_4=-(g(\daleth_2,\daleth_4)\daleth_1-g(\daleth_1,\daleth_4)\daleth_2).$$

This result completes the proof.

Next, we have another important result.

Theorem 2: Let (M,φ,ξ,η,g) be a para-Sasakian manifold admitting curvature condition $Q(\daleth_1,\daleth_2).Ric=0$. Then, we have that either $\psi=-2n$ or the manifold M is an Einstein manifold with the scalar curvature $r=-2n(2n+1)$.

Proof: Let us suppose that the structure (φ,ξ,η,g) of the manifold M satisfies the condition $Q(\daleth_1,\daleth_2).Ric=0$, namely

$$(Q(\daleth_1,\daleth_4).Ric)\daleth_3=0$$

and hence

$$Q(\daleth_1,\daleth_2).Ric\,\daleth_3-Ric\,(Q(\daleth_1,\daleth_2)\daleth_3)=0 \qquad \ldots(31)$$

for all vector fields $\daleth_1,\daleth_2,\daleth_3\epsilon\Gamma(TM)$. If we take $\daleth_1=\xi$ in (31) and also benefit from (19), we obtain

$$\left(1+\frac{\psi}{2n}\right)[R(\xi,\daleth_2)Ric\,\daleth_3-Ric\,(R(\xi,\daleth_2)\daleth_3)]=0,$$

which together with (14) gives

$$\left(1+\frac{\psi}{2n}\right)[\eta(Ric\,\daleth_3)\daleth_2-g(Ric\,\daleth_3,\daleth_2)\xi-Ric\,(\eta(\daleth_3)\daleth_2-g(\daleth_2,\daleth_3)\xi)]=0. \qquad \ldots(32)$$

Due to the equations (11) and (12), the equation (32) reduces to

$$\left(1+\frac{\psi}{2n}\right)[-2n\eta(\daleth_3)\daleth_2 - S(\daleth_2,\daleth_3)\xi - \eta(\daleth_3)Ric\,\daleth_2 - 2ng(\daleth_2,\daleth_3)\xi] = 0. \qquad \ldots(33)$$

Taking inner product of (33) with ξ yields

$$\left(1+\frac{\psi}{2n}\right)[-2ng(\daleth_2,\daleth_3) - S(\daleth_2,\daleth_3)] = 0.$$

Thus, we have that either $\psi = -2n$ or $S(\daleth_2,\daleth_3) = -2ng(\daleth_2,\daleth_3)$. Taking the trace of the ricci tensor, we obtain $r = -2n(2n+1)$. Therefore, we get the requested result.

The next result provides a important characterization regarding para-Sasakian manifolds.

Theorem 3: Let (M,φ,ξ,η,g) be a para-Sasakian manifold admitting curvature condition $Q(\daleth_1,\daleth_2).S = 0$. Then, we have that either $\psi = -2n$ or the manifold is an Einstein manifold.

Proof: Under our hypothesis, we have

$$(Q(\daleth_1,\daleth_2).S)(\daleth_3,\daleth_5) = 0$$

for all vector fields $\daleth_1,\daleth_2,\daleth_3,\daleth_5 \epsilon \Gamma(TM)$. This is equivalent to

$$S(Q(\daleth_1,\daleth_2)\daleth_3,\daleth_5) + S(\daleth_3,Q(\daleth_1,\daleth_2)\daleth_5) = 0. \qquad \ldots(34)$$

Putting $\daleth_1 = \xi$ in (34) and from (19) we get

$$\left(1+\frac{\psi}{2n}\right)(S(R(\xi,\daleth_2)\daleth_3,\daleth_5) + S(\daleth_3,R(\xi,\daleth_2)\daleth_5) = 0. \qquad \ldots(35)$$

Implementing (14) in (35), we find that

$$\left(1+\frac{\psi}{2n}\right)\begin{bmatrix}\eta(\daleth_3)S(\daleth_2,\daleth_5) + 2ng(\daleth_2,\daleth_3)\eta(\daleth_5) + \eta(\daleth_5)S(\daleth_2,\daleth_3)\\ +2ng(\daleth_2,\daleth_5)\eta(\daleth_3)\end{bmatrix} \qquad \ldots(36)$$

Moreover setting $\daleth_3 = \xi$ in (36) provides

$$\left(1+\frac{\psi}{2n}\right)[S(\daleth_2,\daleth_5) + 2ng(\daleth_2,\daleth_5)] = 0,$$

where we have used the equations (5) and (12). Therefore we have the desired result. The proof is completed.

Theorem 4: Let $(M, \varphi, \xi, \eta, g)$ be a para-Sasakian manifold admitting curvature condition $Q.\varphi = 0$. Then we have that $\psi = -2n$.

Proof: Let us consider that the structure (φ, ξ, η, g) of the manifold M satisfies the curvature condition $Q.\varphi = 0$, that is

$$(Q(\daleth_1, \daleth_2).\varphi)\daleth_3 = 0,$$

which is equivalent to

$$Q(\daleth_1, \daleth_2).\varphi\daleth_3 - \varphi(Q(\daleth_1, \daleth_2)\daleth_3) = 0 \qquad \ldots(37)$$

for all vector fields $\daleth_1, \daleth_2, \daleth_3 \epsilon \Gamma(TM)$. Putting ξ instead of $\daleth_1$ in (37), we write

$$Q(\xi, \daleth_2)\varphi\daleth_3 - \varphi(Q(\xi, \daleth_2)\daleth_3) = 0. \qquad \ldots(38)$$

Also, making use of (19) in (38) we get

$$\left(1 + \frac{\psi}{2n}\right)[R(\xi, \daleth_2)\varphi\daleth_3 - \varphi(R(\xi, \daleth_2)\daleth_3] = 0. \qquad \ldots(39)$$

From (2), (3), (14) and (39), we arrive at

$$\left(1 + \frac{\psi}{2n}\right)[-g(\daleth_2, \varphi\daleth_3)\xi - \eta(\daleth_3)\varphi\daleth_2] = 0. \qquad \ldots(40)$$

Replacing $\daleth_3$ by $\varphi\daleth_3$ in (40) and after a straightforward computation, we have

$$\left(1 + \frac{\psi}{2n}\right)[g(\varphi\daleth_2, \varphi\daleth_3)\xi] = 0. \qquad \ldots(41)$$

Taking inner product of (41) with ξ gives

$$\left(1 + \frac{\psi}{2n}\right)[g(\varphi\daleth_2, \varphi\daleth_3)] = 0.$$

Taking the orthonormal basis of the above equation, we obtain that $\psi = -2n$. Therefore, the proof is completed.

The final result of our work is as follows:

Theorem 5: Let $(M, \varphi, \xi, \eta, g)$ be a para-Sasakian manifold admitting curvature condition $\varphi . Q = 0$. Then we have that $\psi = -2n$.

Proof: Let us consider that the structure (φ, ξ, η, g) of the manifold M satisfies the curvature condition $\varphi . Q = 0$, which means

$(\varphi . Q)(\daleth_1, \daleth_2)\daleth_3$

namely,

$$\varphi(Q(\daleth_1, \daleth_2)\daleth_3 - Q(\varphi\daleth_1, \daleth_2)\daleth_3 - Q(\daleth_1, \varphi\daleth_2)\daleth_3 - Q(\daleth_1, \daleth_2)\varphi\daleth_3 = 0 \quad \ldots(42)$$

for all vector fields $\daleth_1, \daleth_2, \daleth_3 \epsilon \Gamma(TM)$. Taking ξ in place of $\daleth_3$ in (42) and it follows from (3) that we have

$$\varphi(Q(\daleth_1, \daleth_2)\xi - Q(\varphi\daleth_1, \daleth_2)\xi - Q(\daleth_1, \varphi\daleth_2)\xi = 0. \quad \ldots \quad (43)$$

Using (13) and (18) in (43) one immediately has

$$\left(1 + \frac{\psi}{2n}\right)(\eta(\daleth_1)\daleth_2 - \eta(\daleth_2)\daleth_1 + \eta(\daleth_2)\varphi\daleth_1 - \eta(\daleth_1)\varphi\daleth_2) = 0. \quad (44)$$

Taking inner product of (44) with arbitrary vector field $\daleth_3$ gives

$$\left(1 + \frac{\psi}{2n}\right)\begin{bmatrix} \eta(\daleth_1)g(\daleth_2, \daleth_3) - \eta(\daleth_2)g(\daleth_1, \daleth_3) \\ +\eta(\daleth_2)g(\varphi\daleth_1, \daleth_3) - \eta(\daleth_1)g(\varphi\daleth_2, \daleth_3) \end{bmatrix} = 0. \quad \ldots \quad (45)$$

Setting $\daleth_1 = \daleth_3 = G_i$ in (45) and from (2), (3) we get

$$\left(1 + \frac{\psi}{2n}\right)\eta(\daleth_2) = 0,$$

where $\{G_i\}_{i=1}^{2n+1}$ is a local orthonormal frame of the manifold M. Taking $\daleth_2 = \xi$ yields $\psi = -2n$. Therefore, the proof is completed.

REFERENCES

Acet, B. E., Kılıç, E., & Yüksel Perktaş, S. (2012). Some curvature conditions on a Para-Sasakian manifold with canonical paracontact connection, *International Journal of Mathematics and Mathematical Sciences*, 2012, 1-24.

Adati T. & Matsumoto, K. (1977). On conformally recurrent and conformally symmetric P-Sasakian manifolds, *TRU Math.*,13, 25-32.

Adati, T. Miyazawa, T. (1979), On P-Sasakian manifolds satisfying certain conditions, *Tensor (N.S.)* 33, 173–178.

Bağdatli Yilmaz, H. (2020). Sasakian manifolds satisfying certain conditions Q tensor, *Journal of Geometry*, 111, 1-10.

De, U. C. & Guha, N. (1992). On a type of P-Sasakian manifold, *Istanbul Univ. Fen Fak. Mat. Der.*, 51, 35-39.

De, U. C., Ozgür, C., Arslan, K., Murathan, C. & Yildiz, A. (2008). On a type of P-Sasakian manifolds, *Mathematica Balkanica*, 22, 25–36.

De, U.C., Han, Y. & Mandal, K. (2017), On Para-Sasakian manifolds satisfying certain curvature conditions, *Filomat*, 31 (7), 1941-1947.

De U. C. & Majhi, P. (2019). On the Q Curvature tensor of a generalized Sasakian-space-form, *Kragujevac Journal of Mathematics*, 43 (3), 333-349.

Deshmukh, S. & Ahmed, S. (1980). Para-Sasakian manifolds isometrically immersed in spaces of constant curvature, *Kyungpook J. Math.* 20, 112–121.

Mantica, C. A., Suh Y. J. (2013). Pseudo-Q-symmetric Riemannian manifolds, *International. Journal of Gemetric Methods in Modern Physics,* 10 (5).

Matsumoto, K., Ianus, S. & Mihai, I. (1986). On P-Sasakian manifolds which admit certain tensor fields, *Publ. Math. Debrecen* 33, 199-204.

Özgür, C. (2005). On a class of Para-Sasakian manifolds, *Turkish J. Math*., 29, 249–257.

Özgür, C. Tripathi, M.M. (2007). On P-Sasakian manifolds satisfying certain conditions on the concircular curvature tensor, *Turkish J. Math.*, 31, 171–179.

Sato, I. (1976). On a structure similar to the almost contact structure, *Tensor, N.S.*, 30, 219-224.

Sharfuddin, A., Deshmukh, S.& Husain, S.I. (1980). On Para-Sasakian manifolds, *Indian J. pure appl. Math.*, 11, 845–853.

Yadav S. K & Yıldız, A. (2022). Q-curvature tensor on f-Kenmotsu $3-$Manifolds, *Universal Journal of Mathematics and Applications*, 5 (3), 96-106.

Yıldırım, M. (2022). A new characterization of Kenmotsu manifolds with respect to Q tensor, *Journal of Geometry and Physics*, 176.

Zamkovoy, S. (2009) Canonical connections on paracontact manifolds, *Annals of Global Analysis and Geometry*, 36 (1), 37–60.

CHAPTER III

Exploring Bicomplex Lebesgue Spaces: Properties and Significance

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Introduction

Lebesgue spaces are essential structures in functional analysis and measure theory, offering a formal framework for understanding function convergence and integrability. The study of Lebesgue spaces has been helpful in increasing our understanding of diverse mathematical phenomena, providing a flexible platform for examining the behavior of functions in various circumstances. As we explore deeper into the complexities of Lebesgue spaces, this essay focuses on the concepts of sums and intersection within this mathematical realm. We want to get insights into the convergence

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features and interrelationships that emerge when combining or intersecting functions in Lebesgue spaces by investigating the interaction of these operations. This examination not only advances theoretical comprehension of these spaces, but also reveals their practical consequences in a variety of mathematical applications. Join us on a voyage through the intricacies of sums and crossings in Lebesgue spaces, where the convergence of mathematical concepts leads to a better knowledge of function structure and behavior $\mathbb{BC}$-valued functions arise naturally in various mathematical fields, including probability theory, mathematical analysis, and functional analysis, and understanding their properties is crucial for advancing these areas of study. Indeed, the study of modules with bicomplex scalars in the context of functional analysis has gained significant attention in recent years. One influential work that has contributed to this area is the book (Alpay et al., 2014). The book likely presents groundbreaking results and insights related to this topic. Functional analysis traditionally deals with vector spaces over a field, such as the complex numbers or the real numbers. However, by considering modules with bicomplex scalars, where the scalars are elements of the bicomplex numbers, a broader framework is introduced. This extension allows for the exploration of new mathematical structures and the investigation of properties beyond the classical setting. The book by Alpay et al. is likely a valuable resource for researchers and enthusiasts interested in this area. It likely presents notable results, techniques, and applications pertaining to the study of modules with bicomplex scalars in the context of functional analysis. These results may encompass various aspects of functional analysis, such as operator theory, function spaces, and spectral theory, among others. They may shed light on the behavior of modules with bicomplex scalars, reveal connections to other areas of mathematics, and potentially find applications in physics, engineering, or other disciplines.

The series of articles mentioned in the references highlight the systematic study of topological bicomplex modules and various fundamental theorems related to them. Here is a breakdown of the articles and their contributions:

In (Kumar & Saini, 2016), the authors studied topological bicomplex modules, likely exploring their topological properties and investigating concepts such as convergence, continuity, and compactness in this context.

The authors in (Kumar, Kumar & Rochon, 2011) presented fundamental theorems, including Banach-Steinhaus theorem, open mapping theorem, closed graph theorem and interior mapping theorem for bicomplex modules.

The papers (Saini, Sharma & Kumar, 2020), in collaboration with (Kumar, Kumar & Rochon, 2011), likely extends the study of fundamental theorems to the setting of topological bicomplex modules. The focus may be on generalizing classical results from functional analysis to the bicomplex module framework, providing a deeper understanding of their properties. Also, the authors likely delve further into the study of topological hyperbolic modules, topological bicomplex modules, exploring the properties of linear operators, continuity, and related topological concepts specific to these settings.

The authors in (Luna- Elizarrarás, Perez-Regalado & Shapiro, 2014) studied on bicomplex modules and hyperbolic modules and wrote the Hahn-Banach theorem for these modules.

The book (Luna-Elizarrarás et al., 2015) likely provides an in-depth exploration of bicomplex analysis and geometry. It may cover a wide range of topics, including holomorphic functions, integration, differential equations, and geometric properties specific to the bicomplex domain.

In (Colombo, Sabatini & Struppa, 2014), the authors focused on $\mathbb{BC}$ bounded linear operators and bicomplex functional calculus. It may provide a detailed study of operators acting on bicomplex modules and explore the construction and properties of functional calculi specific to the bicomplex framework.

In (Sağır, Değirmen & Duyar, 2023), properties of bicomplex matrix transformations between sequences' spaces c_0 and c are examined.

These references collectively represent significant contributions to the study of bicomplex modules, functional analysis, and related areas. They showcase the exploration of properties, the development of new theorems, and the application of functional analysis techniques in the context of bicomplex numbers. Researchers and readers interested in these topics can refer to these articles and the books for detailed insights into the respective areas of study. Now, we will give a small summary of bicomplex numbers with some basic properties.

Preliminaries on $\mathbb{BC}$ and $\mathbb{BC}$ -Lebesgue spaces

The set bicomplex numbers $\mathbb{BC}$ which is a four-dimensional extension of the complex numbers is defined as

$$\mathbb{BC} := \{W = w_1 + jw_2 |\, w_1,\ w_2 \in \mathbb{C}(i)\}.$$

Here i and j are imaginary units satisfying $ij = ji$ and $i^2 = -1 = j^2$. Here $\mathbb{C}(i)$ with the imaginary unit i, stands for the field of complex numbers. According to ring structure, for any $Z = z_1 + jz_2$, $W = w_1 + jw_2$ in $\mathbb{BC}$ usual addition and multiplication are defined as

$$Z + W = (z_1 + w_1) + j(z_2 + w_2)$$

$$ZW = (z_1w_1 - z_2w_2) + j(z_2w_1 + z_1w_2).$$

Under the ordinary addition and multiplication of bicomplex numbers, the set $\mathbb{BC}$ forms a commutative ring. The bicomplex numbers have a unit element denoted as $1_{\mathbb{BC}} := 1$ and this acts as the identity for multiplication, such that for any bicomplex number W, $1 \cdot W = W \cdot 1 = W$. In the sense of module structure, the set $\mathbb{BC}$ is a module over itself. This means that $\mathbb{BC}$ satisfies the properties of a module, including scalar multiplication and distributivity. If one multiplies the imaginary units i and j, then a new hyperbolic unit k can be obtained such that $k^2 = 1$. This implies that k is a square root of 1 and is distinct from i and j. The product operation of all units i, j and k in the bicomplex numbers is commutative. Specifically, the following relations hold:

$$ij = k,\ jk = -i\ \ and\ \ ik = -j.$$

These properties summarize the basic characteristics of bicomplex numbers and their algebraic structure.

Hyperbolic numbers $\mathbb{D}$ are a two-dimensional extension of the real numbers that form a number system known as the hyperbolic plane or hyperbolic plane algebra. They can be represented in the form $\alpha = \beta_1 + k\beta_2$, where β_1 and β_2 are real numbers, and k is the hyperbolic unit. In the hyperbolic number system, for any two hyperbolic numbers $\alpha = \beta_1 + k\beta_2$ and $\gamma = \delta_1 + k\delta_2$, addition and multiplication are defined as follows:

$$\alpha + \gamma = (\beta_1 + \delta_1) + k(\beta_2 + \delta_2)$$

$$\alpha\gamma = (\beta_1\delta_1 + \beta_2\delta_2) + k(\beta_1\delta_2 + \beta_2\delta_1).$$

The hyperbolic numbers form a ring, however, unlike the complex numbers, the hyperbolic numbers do not have a multiplicative inverse for all nonzero elements. The nonzero hyperbolic numbers that have multiplicative inverses are called units. The bicomplex numbers contain two imaginary units i and j, and the hyperbolic numbers can be taken as a subset of the bicomplex numbers by restricting the imaginary part of j to be zero.

Let $W = w_1 + jw_2 \in \mathbb{BC}$ where $w_1,\ w_2 \in \mathbb{C}(i)$. By the notation of W with imaginary units i and j, three conjugations are brought out for bicomplex numbers in (Alpay et al., 2014) and (Luna-Elizarrarás et al., 2015) as $\bar{W}_1 = \overline{w_1} + j\overline{w_2},\ \bar{W}_2 = w_1 - jw_2$

and $\overline{W}_3 = \overline{w_1} - j\overline{w_2}$ where $\overline{w_1}$ and $\overline{w_2}$ are the usual complex conjugates of $w_1, w_2 \in \mathbb{C}(i)$. For any bicomplex number W, they also wrote the following three moduli in (Alpay et al., 2014), (Luna-Elizarrarás et al., 2015) and (Price, 2018) as:

i. $|W|_i^2 = W \cdot \overline{W}_2 = {w_1}^2 + {w_2}^2 \in \mathbb{C}(i)$,

ii. $|W|_j^2 = W \cdot \overline{W}_1 = (|w_1|^2 - |w_2|^2) + j\big(2Re(w_1\overline{w_2})\big) \in \mathbb{C}(j)$

iii. $|W|_k^2 = W \cdot \overline{W}_3 = (|w_1|^2 + |w_2|^2) + k\big(-2Im(w_1\overline{w_2})\big) \in \mathbb{D}$.

Furthermore, $\mathbb{BC}$ is a normed space with the norm

$$\|W\|_{\mathbb{BC}} = \sqrt{|w_1|^2 + |w_2|^2}$$

for any $W = w_1 + jw_2$ in $\mathbb{BC}$ (Alpay et al., 2014). According to this,

$$\|W_1W_2\|_{\mathbb{BC}} \leq \sqrt{2}\|W_1\|_{\mathbb{BC}}\|W_2\|_{\mathbb{BC}}$$

for every W_1, $W_2 \in \mathbb{BC}$, and finally $\mathbb{BC}$ is a quasi-Banach algebra (Alpay et al., 2014). If the hyperbolic numbers e_1 and e_2 defined as

$$e_1 = \frac{1+k}{2} \quad \text{and} \quad e_2 = \frac{1-k}{2},$$

then it is easy to see that the set $\{e_1, e_2\}$ is a fundamental set in $\mathbb{C}(i)$-vector space $\mathbb{BC}$ and linearly independent. The set $\{e_1, e_2\}$ also satisfies the following properties:

$$e_1^2 = e_1, \qquad e_2^2 = e_2, \qquad (\overline{e_1})_3 = e_1, \ (\overline{e_2})_3 = e_2$$

$$e_1 + e_2 = 1, \quad e_1 \cdot e_2 = 0$$

with $\|e_1\|_{\mathbb{BC}} = \|e_2\|_{\mathbb{BC}} = \frac{\sqrt{2}}{2}$. By using this linearly independent set $\{e_1, e_2\}$, any $W = w_1 + jw_2 \in \mathbb{BC}$ can be written as a linear combination of e_1 and e_2 uniquely. That is, $W = w_1 + jw_2$ can be written as

$$W = w_1 + jw_2 = e_1 z_1 + e_2 z_2 \quad (1.1)$$

where $z_1 = w_1 - iw_2$ and $z_2 = w_1 + iw_2$ (Alpay et al., 2014). Here z_1 and z_2 are elements of $\mathbb{C}(i)$ and (1.1), the preceding formula, is named with the *idempotent representation* of W.

Besides the Euclidean-type norm $\|\cdot\|_{\mathbb{BC}}$, another norm named with ($\mathbb{D}$-valued) hyperbolic-valued norm $|W|_k$ of any bicomplex number $W = e_1 z_1 + e_2 z_2$ is defined as

$$|W|_k = e_1 |z_1| + e_2 |z_2|.$$

For any hyperbolic number $\alpha = \beta_1 + k\beta_2 \in \mathbb{D}$, an idempotent representation can also be written as

$$\alpha = e_1 \alpha_1 + e_2 \alpha_2$$

where $\alpha_1 = \beta_1 + \beta_2$ and $\alpha_2 = \beta_1 - \beta_2$ are real numbers. If $\alpha_1 > 0$ and $\alpha_2 > 0$ for any $\alpha = \beta_1 + k\beta_2 \in \mathbb{D}$, then we say that α is called a positive hyperbolic number. Thus, $\mathbb{D}^+ \cup \{0\}$, the set of non-negative hyperbolic numbers

$$\begin{aligned} \mathbb{D}^+ \cup \{0\} &= \{\alpha = \beta_1 + k\beta_2 : \ \beta_1^2 - \beta_2^2 \geq 0, \beta_1 \geq 0\} \\ &= \{\alpha = e_1\alpha_1 + e_2\alpha_2 : \ \alpha_1, \alpha_2 \geq 0\}. \end{aligned}$$

can be defined. Now, let α and γ be any two elements of $\mathbb{D}$. In (Alpay et al., 2014) and (Luna-Elizarrarás et al., 2015), a relation $\preccurlyeq$ is defined on $\mathbb{D}$ by

$$\alpha \preccurlyeq \gamma \quad \Leftrightarrow \gamma - \alpha \in \mathbb{D}^+ \cup \{0\}.$$

It is showed in (Alpay et al., 2014) that this relation "$\preccurlyeq$" has reflexive, anti-symmetric and transitive properties. Therefore "$\preccurlyeq$"

can define a partial order relation on $\mathbb{D}$. If idempotent representations of the hyperbolic numbers α, γ are written as $\alpha = e_1\alpha_1 + e_2\alpha_2$ and $\gamma = e_1\gamma_1 + e_2\gamma_2$, then $\alpha \preccurlyeq \gamma$ implies that $\alpha_1 \leq \gamma_1$ and $\alpha_2 \leq \gamma_2$. By $\alpha \prec \gamma$, we mean $\alpha_1 < \gamma_1$ and $\alpha_2 < \gamma_2$. For more details on hyperbolic numbers $\mathbb{D}$ and partial order "$\preccurlyeq$", one can refer to Section 1.5 of (Alpay et al., 2014), (Luna-Elizarrarás et al., 2015) and (Price, 2018).

Definition 1. Let U be a subset of $\mathbb{D}$. U is called a $\mathbb{D}$-bounded above set if there is a hyperbolic number δ such that $\delta \succcurlyeq \alpha$ for all $\alpha \in U$. If $U \subset \mathbb{D}$ is a $\mathbb{D}$-bounded set from above, then the hyperbolic supremum of U is defined as the smallest member of the set of all upper bounds of U (Luna-Elizarrarás et al., 2015).

In other words, the hyperbolic number $\lambda = e_1\lambda_1 + e_2\lambda_2$, where $\lambda_1, \lambda_2 \in \mathbb{R}$, is the $\mathbb{D}$-supremum of U if

- **i.** $e_1\alpha_1 + e_2\alpha_2 \preccurlyeq e_1\lambda_1 + e_2\lambda_2$ for each $\alpha = e_1\alpha_1 + e_2\alpha_2 \in U$
- **ii.** For any $\varepsilon = e_1\varepsilon_1 + e_2\varepsilon_2 \succ 0$, there exists $\theta = e_1\theta_1 + e_2\theta_2 \in U$ such that $e_1\theta_1 + e_2\theta_2 \succ e_1(\lambda_1 - \varepsilon_1) + e_2(\lambda_2 - \varepsilon_2)$

are satisfied.

Remark 1. Let $e_1\theta_1 + e_2\theta_2$ be a $\mathbb{D}$-bounded above subset of $\mathbb{D}$ and $U_1 := \{\gamma_1 : e_1\gamma_1 + e_2\gamma_2 \in U\}$, $U_2 := \{\gamma_2 : e_1\gamma_1 + e_2\gamma_2 \in U\}$. Then the $sup_{\mathbb{D}}U$ is given by

$$sup_{\mathbb{D}}U := e_1 supU_1 + e_2 supU_2.$$

Similarly, for any set U which is $\mathbb{D}$-bounded from below, $\mathbb{D}$-infimum of U can be defined as

$$inf_{\mathbb{D}}U := e_1 infU_1 + e_2 infU_2$$

where U_1 and U_2 are as above, [Remark 1.5.2] (Alpay et al., 2014).

Definition 2. Let $(X,+)$ be an abelian group and $(X,+,\cdot)$ be a $\mathbb{BC}$-module. If there is a topology τ_X in X, such that the operations $+: X \times X \to X$ and $\cdot: \mathbb{BC} \times X \to X$ are continuous, then $(X,+,\cdot)$ is called a topological $\mathbb{BC}$-module.

Remark 2. A $\mathbb{BC}$ -module space or $\mathbb{D}$-module space Y can be decomposed as

$$Y = e_1Y_1 + e_2Y_2 \tag{1.2}$$

where $Y_1 = e_1Y$ and $Y_2 = e_2Y$ are $\mathbb{R}$-vector or $\mathbb{C}(i)$-vector spaces. The spelling in (1.2) is called as the idempotent decomposition of the space Y. Therefore, any element y in Y can be uniquely inscribed as $y = e_1y_1 + e_2y_2$ with $y_1 \in Y_1$ and $y_2 \in Y_2$, (Alpay et al., 2014).

The following is known from (Saini, Sharma & Kumar, 2020).

Definition 3. Let $(X, \|\cdot\|_X)$ be a $\mathbb{BC}$-module. If every Cauchy sequence in X converges to any element of it with respect to the norm, then $(X, \|\cdot\|_X)$ is called a bicomplex Banach module.

Proposition 1. $(X, \|\cdot\|_X)$is a bicomplex Banach module, if and only if, the decomposition pairs of the space, $\left(X_1, \|\cdot\|_{X_1}\right)$ and $\left(X_2, \|\cdot\|_{X_2}\right)$ are complex Banach spaces (Kumar & Saini, 2016).

Definition 4. Let $T: X \to X$ be a map. Then T is called a $\mathbb{BC}$-linear operator on X, if the following exist:

i. $T(x+y) = T(x) + T(y)$,

ii. $T(\alpha x) = \alpha T(x)$

for every $x, y \in X$ and $\alpha \in \mathbb{BC}$.

The following result is well known from (Colombo, Sabatini & Struppa, 2014).

Proposition 2. Let X be a bicomplex Banach module and $T: X \to X$ be a linear operator. Suppose that X has an idempotent decomposition as $X = e_1X_1 + e_2X_2$. Then the operator T admits the idempotent representation $T = e_1T_1 + e_2T_2$, where

$$T_j: e_jX \rightarrow e_jX$$

$$x \rightarrow T_j(x) := e_jT(e_jx)$$

are linear operators for $j = 1,2$ respectively.

Definition 5. Let $\mathcal{M}$ be a σ-algebra on a set Ω and $\mu = \mu_1e_1 + \mu_2e_2$ be a bicomplex-valued function defined on Ω. Then μ is called a bicomplex measure on $\mathcal{M}$ if μ_1 and μ_2 are both complex measures on $\mathcal{M}$. Nevertheless if μ_1 and μ_2 are positive measures on $\mathcal{M}$ namely, range of both μ_1, μ_2 are $[0,\infty]$, then μ is called a $\mathbb{D}$-measure on $\mathcal{M}$. Also, μ is called a $\mathbb{D}^+$-measure on $\mathcal{M}$, if μ_1, μ_2 are real measures on $\mathcal{M}$ i.e. $\mu_1(\cdot), \mu_2(\cdot) \in [0,\infty)$, (Ghosh & Mondal, 2022).

Assume that $\Omega = (\Omega, \mathcal{M}, \mu)$ is a σ-finite complete measure space and f_1, f_2 are complex-valued (real-valued) measurable functions on Ω. The function having idempotent decomposition $f = f_1e_1 + f_2e_2$ is called as a $\mathbb{BC}$-measurable function and $|f|_k = |f_1|e_1 + |f_2|e_2$ is called a $\mathbb{D}$-valued measurable function on Ω (Dubey, Kumar & Sharma, 2014). Thus, for any given complex valued function space $(F(\Omega), \|\cdot\|_\Omega)$, one can create a $\mathbb{BC}$-valued function space $(F(\Omega, \mathbb{BC}), \|\cdot\|_{\mathbb{BC}})$ by combining all f_1, f_2 and bringing out functions of the type

$$f = f_1e_1 + f_2e_2$$

where f_1 and f_2 are in $(F(\Omega), \|\cdot\|_\Omega)$ with $\|f\|_{\mathbb{BC}}^2 = \frac{1}{2}(\|f_1\|_\Omega^2 + \|f_2\|_\Omega^2)$. Similar definition can be given for any hyperbolic measurable function.

For any $\mathbb{BC}$-valued measurable function $f = f_1e_1 + f_2e_2$, it is easy to see that $|f|_k = |f_1|e_1 + |f_2|e_2$ is $\mathbb{D}$-valued measurable. Because if $f = f_1e_1 + f_2e_2$ is a $\mathbb{BC}$-valued measurable function, then f_1 and f_2 are $\mathbb{C}$-measurable functions. Therefore real and imaginary parts of f_1 and f_2 are $\mathbb{R}$-valued measurable and so does $|f_1|$ and $|f_2|$. As a result, $|f|_k$ is $\mathbb{D}$-measurable. Also, for any two $\mathbb{BC}$-valued measurable functions f and g, it can be easily seen that their sum and multiplication functions are also $\mathbb{BC}$-measurable functions, (Dubey, Kumar & Sharma, 2014) and (Ghosh & Mondal, 2022). More results on $\mathbb{D}$-topology such as $\mathbb{D}$-limit, $\mathbb{D}$-continuity, $\mathbb{D}$-Cauchy and $\mathbb{D}$-convergence etc. can be found in (Değirmen & Sağır, 2023), (Ghosh & Mondal, 2022), (Toksoy & Sağır, 2023) and the references therein.

Definition 6. Let $\mathcal{M}$ be a σ-algebra and $\mu = e_1\mu_1 + e_2\mu_2$ be a $\mathbb{BC}$-measure on $(\Omega, \mathcal{M})$. Then any two bicomplex valued $\mathbb{BC}$-measurable functions $f = e_1 f_1 + e_2 f_2$ and $g = e_1 g_1 + e_2 g_2$ on Ω are called to be equal (μ-a.e.) if $f_1 = g_1$ (μ_1-a.e.) and $f_2 = g_2$ (μ_2-a.e.).

Definition 7. Let $\mu = e_1\mu_1 + e_2\mu_2$ be a $\mathbb{D}$-measure on a measure space $(\Omega, \mathcal{M})$ and $1 \leq p < \infty$. Suppose $L^p(\Omega, \mu_1)$ and $L^p(\Omega, \mu_2)$ stand for the linear space of all (equivalence classes of) complex valued, measurable functions f_1 and f_2 defined on Ω with

$$\int_\Omega |f_1(x)|^p d\mu_1 < \infty \quad \text{and} \quad \int_\Omega |f_2(x)|^p d\mu_2 < \infty.$$

Then the space $L^p_{\mathbb{BC}}(\Omega, \mathcal{M}, \mu) = L^p_{\mathbb{BC}}(\mu)$ consist of equivalence classes all bicomplex valued, bicomplex measurable functions $f = e_1 f_1 + e_2 f_2$ on Ω such that $f_1 \in L^p(\Omega, \mu_1)$ and $f_2 \in L^p(\Omega, \mu_2)$ (Toksoy & Sağır, 2023).

Proposition 3. For $1 \leq p < \infty$, $L^p_{\mathbb{BC}}(\mu)$ is a $\mathbb{BC}$-module under usual addition operation in functions and bicomplex scalar multiplication (Toksoy & Sağır, 2023).

Let $1 \leq p < \infty$. By using Definition 2 and Remark 2, we may write an idempotent decomposition

$$L^p_{\mathbb{BC}}(\mu) = e_1 L^p(\mu_1) + e_2 L^p(\mu_2)$$

for $L^p_{\mathbb{BC}}(\mu)$ where $L^p(\mu_1)$ and $L^p(\mu_2)$ are usual Lebesgue spaces, (Toksoy & Sağır, 2023). Therefore, a hyperbolic ($\mathbb{D}$-valued) norm can be defined on the $\mathbb{BC}$-module $L^p_{\mathbb{BC}}(\mu)$ with

$$\|f\|_{p,\mathbb{D}} = e_1\|f_1\|_{p,\mu_1} + e_2\|f_2\|_{p,\mu_2}$$

for any $e_1 f_1 + e_2 f_2 = f \in L^p_{\mathbb{BC}}(\mu)$.

Proposition 4. The space $\left(L^p_{\mathbb{BC}}(\mu), \|\cdot\|_{p,\mathbb{D}}\right)$ is a bicomplex Banach module for $1 \le p < \infty$ (Toksoy & Sağır, 2023).

In (Toksoy & Sağır, 2023), by using Definition 2.2 of (Değirmen & Sağır, 2023), a new functional

$$\|f\|_{p,k} = \left(\int_\Omega |f(x)|_k^p d\mu\right)^{\frac{1}{p}}$$

$$= \left(\int_\Omega |e_1 f_1(x) + e_2 f_2(x)|_k^p (e_1 d\mu_1 + e_2 d\mu_2)\right)^{\frac{1}{p}} \qquad (1.3)$$

is defined and showed that $\|f\|_{p,\mathbb{D}} = \|f\|_{p,k}$ for any $f \in L^p_{\mathbb{BC}}(\mu)$.

New Results on $\mathbb{BC}$-Lebesgue spaces

Proposition 5. Let $1 \le p < \infty$. The set

$$\mathbb{S} = \{s = s_1 e_1 + s_2 e_2 \mid s_1,\ s_2 \in S\}$$

is $\mathbb{D}-$dense in $L^p_{\mathbb{BC}}(\Omega, \mathcal{M}, \mu)$ where S is the set of simple functions.

Proof. Let $\varepsilon = e_1\varepsilon_1 + e_2\varepsilon_2 \succ 0$ and $f = e_1 f_1 + e_2 f_2$ be any element of $L^p_{\mathbb{BC}}(\mu)$. By the definition of $L^p_{\mathbb{BC}}(\mu)$, the functions f_1 and f_2 belong to $L^p(\mu_1)$ and $L^p(\mu_2)$. Since the set of simple (step) functions S is dense subset of $L^p(\mu_1)$ and $L^p(\mu_2)$, then there exist simple functions h_1 and h_2 such that

$$\|f_1 - h_1\|_{p,\mu_1} < \varepsilon_1 \quad \text{and} \quad \|f_2 - h_2\|_{p,\mu_2} < \varepsilon_2.$$

If one call $e_1 h_1 + e_2 h_2$ as h, then $h \in \mathbb{S}$ and

$$\|f-h\|_{p,\mathbb{D}} = e_1\|f_1-h_1\|_{p,\mu_1} + e_2\|f_2-h_2\|_{p,\mu_2} \\ \prec e_1\varepsilon_1 + e_2\varepsilon_2 = \varepsilon.$$

This means $\mathbb{S}$ is $\mathbb{D}$ −dense in $L^p_{\mathbb{BC}}(\mu)$.

Remark 3. If we define $C_c(\Omega, \mathbb{BC})$ as the set of all functions $f_1e_1 + f_2e_2$ where $f_1, f_2 \in C_c(\Omega)$, then $C_c(\Omega, \mathbb{BC})$ is $\mathbb{D}$ −dense in $L^p_{\mathbb{BC}}(\mu)$ by Lusin's theorem where $C_c(\Omega)$ is the set of all continuous complex functions on Ω whose support is compact.

The following theorem is Theorem 2.6 of (Toksoy & Sağır, 2023).

Theorem 1. (*Hölder's inequality*) Let $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $f \in L^p_{\mathbb{BC}}(\mu)$, $g \in L^q_{\mathbb{BC}}(\mu)$ with $f = f_1e_1 + f_2e_2$, $g = g_1e_1 + g_2e_2$. Then $fg \in L^1_{\mathbb{BC}}(\mu)$ and

$$\|fg\|_{1,\mathbb{D}} \preccurlyeq \|f\|_{p,\mathbb{D}}\|g\|_{q,\mathbb{D}}.$$

Theorem 2. Let f be an element of $L^p_{\mathbb{BC}}(\Omega, \mathfrak{M}, \mu)$ for $1 < p < \infty$. Then

$$\|f\|_{p,\mathbb{D}} = \sup_{g\in L^q_{\mathbb{BC}}(\mu)}{}_{\mathbb{D}} \left\{\frac{\|fg\|_{1,\mathbb{D}}}{\|g\|_{q,\mathbb{D}}} : g \neq 0,\ \frac{1}{p} + \frac{1}{q} = 1\right\}.$$

Proof. Using Hölder's inequality and (1.3), we can write that

$$\|fg\|_{1,\mathbb{D}} = \|fg\|_{1,k} = \int_\Omega |fg|_k d\mu \preccurlyeq \|f\|_{p,\mathbb{D}}\|g\|_{q,\mathbb{D}}.$$

Then

$$\|fg\|_{1,\mathbb{D}}\|g\|^{-1}_{q,\mathbb{D}} \preccurlyeq \|f\|_{p,\mathbb{D}}$$

for all $g \neq 0$ and this implies that

$$\sup_{g\in L^q_{\mathbb{BC}}(\mu)}{}_{\mathbb{D}} \left\{\frac{\|fg\|_{1,\mathbb{D}}}{\|g\|_{q,\mathbb{D}}} : g \neq 0,\ \frac{1}{p} + \frac{1}{q} = 1\right\} \preccurlyeq \|f\|_{p,\mathbb{D}}. \tag{1.4}$$

Now suppose that f is non-zero and $g = \beta|f|_k^{p-1}$ where β is a constant. In that case $|fg|_k = |\beta|_k|f|_k^p$ and $\|fg\|_{1,\mathbb{D}} = |\beta|_k\|f\|_{p,k}^p$. If one chooses $|\beta|_k$ as $\|f\|_{p,\mathbb{D}}^{1-p}$, then the equality

$$\|fg\|_{1,\mathbb{D}} = |\beta|_k\|f\|_{p,k}^p = \|f\|_{p,\mathbb{D}}^{1-p}\|f\|_{p,\mathbb{D}}^p = \|f\|_{p,\mathbb{D}} \tag{1.5}$$

is written. Since $|g|_k^q = |\beta|_k^q|f|_k^{(p-1)q}$ and $(p-1)q = q$, if we integrate the both sides of this equality, then

$$\|g\|_{q,\mathbb{D}} = \|g\|_{q,k} = \left(\int_\Omega |g|_k^q d\mu\right)^{\frac{1}{q}} = \left(\int_\Omega |\beta|_k^q|f|_k^{(p-1)q} d\mu\right)^{\frac{1}{q}}$$
$$= |\beta|_k\|f\|_{p,k}^{\frac{p}{q}} = \|f\|_{p,\mathbb{D}}^{1-p}\|f\|_{p,k}^{\frac{p}{q}} = 1$$

and $\|g\|_{q,\mathbb{D}}^{-1} = 1$ are obtained. Thus by using (1.5), we can write

$$\|f\|_{p,\mathbb{D}} = \|fg\|_{1,\mathbb{D}}\|g\|_{q,\mathbb{D}}^{-1} \preccurlyeq \sup_{g\in L_{\mathbb{BC}}^q(\mu)}{}_{\mathbb{D}} \left\{\|fg\|_{1,\mathbb{D}}\|g\|_{q,\mathbb{D}}^{-1} : g \neq 0,\ \frac{1}{p} + \frac{1}{q} = 1\right\}. \tag{1.6}$$

Combining this (1.6) with (1.4), we get the result.

The following theorem, Minkowski inequality for $\mathbb{BC}$ −Lebesgue space, is Theorem 2.8 of (Toksoy & Sağır, 2023).

Theorem 3. Let $f = f_1e_1 + f_2e_2$ and $g = g_1e_1 + g_2e_2$ be any two elements of $L_{\mathbb{BC}}^p(\mu)$. Then

$$\|f + g\|_{p,\mathbb{D}} \preccurlyeq \|f\|_{p,\mathbb{D}} + \|g\|_{p,\mathbb{D}}$$

for all $1 \leq p \leq \infty$.

To show the duality of $\mathbb{BC}$-Lebesgue spaces for $1 < p < \infty$, following similar arguments are adapted from the book (Castillo & Rafeiro, 2016).

Theorem 4. Each function $g \in L_{\mathbb{BC}}^q(\mu)$ defines a linear functional Fwhich is $\mathbb{D}$-bounded in $L_{\mathbb{BC}}^p(\mu)$ given by

$$F(f)=\int_{\Omega} fg d\mu$$

and $\|F\|=\|g\|_{q,\mathbb{D}}$.

Proof. Let $g=e_1g_1+e_2g_2$ be a fixed function in $L^q_{\mathbb{BC}}(\mu)$. We will show that F given by

$$F(f)=\int_{\Omega} fg d\mu$$

is a linear functional in $L^p_{\mathbb{BC}}(\Omega,\mathcal{M},\mu)$. Indeed, let α and β be bicomplex numbers and f, h be elements of $L^p_{\mathbb{BC}}(\Omega,\mathcal{M},\mu)$ where $f=e_1f_1+e_2f_2$ and $h=e_1h_1+e_2h_2$. Then

$$F(\alpha f+\beta h)=\int_{\Omega}(\alpha f+\beta h) g d\mu$$

$$=\int_{\Omega}[(\alpha_1f_1+\beta_1h_1)e_1+(\alpha_2f_2+\beta_2h_2)e_2](e_1g_1+e_2g_2)d\mu$$

$$=e_1\int_{\Omega}(\alpha_1f_1+\beta_1h_1)g_1d\mu_1+e_2\int_{\Omega}(\alpha_2f_2+\beta_2h_2)g_2d\mu_2$$

$$=e_1\int_{\Omega}\alpha_1f_1g_1d\mu_1+e_1\int_{\Omega}\beta_1h_1g_1d\mu_1+e_2\int_{\Omega}\alpha_2f_2g_2d\mu_2+e_2\int_{\Omega}\beta_2h_2g_2d\mu_2$$

$$=e_1\int_{\Omega}\alpha_1f_1g_1d\mu_1+e_2\int_{\Omega}\alpha_2f_2g_2d\mu_2+e_1\int_{\Omega}\beta_1h_1g_1d\mu_1+e_2\int_{\Omega}\beta_2h_2g_2d\mu_2$$
$$=(e_1\alpha_1+e_2\alpha_2)\int_{\Omega}(e_1f_1+e_2f_2)(e_1g_1+e_2g_2)(e_1d\mu_1+e_2d\mu_2)$$
$$+(e_1\beta_1+e_2\beta_2)\int_{\Omega}(e_1f_1+e_2f_2)(e_1g_1+e_2g_2)(e_1d\mu_1+e_2d\mu_2)$$
$$=\alpha\int_{\Omega}fgd\mu+\beta\int_{\Omega}hgd\mu$$
$$=\alpha F(f)+\beta F(h).$$

On the other hand

$$|F(f)|_k=\left|\int_{\Omega} fg d\mu\right|_k \preccurlyeq \int_{\Omega}|fg|_k d\mu \preccurlyeq \|f\|_{p,\mathbb{D}}\|g\|_{q,\mathbb{D}}$$

by Theorem 1. Then it can be written that

$$\frac{|F(f)|_k}{\|f\|_{p,\mathbb{D}}} \preccurlyeq \|g\|_{q,\mathbb{D}}$$

which is meaning

$$\|F\| \preccurlyeq \|g\|_{q,\mathbb{D}}. \tag{1.7}$$

This inequality and the previous ones show that F is a $\mathbb{D}$ −bounded linear operator. Furthermore, if we define a function f as

$$f = |g|_k^{q-2}\bar{g}_3 \tag{†}$$

where $\bar{g}_3$ is the third conjugate of g, then

$$fg = |g|_k^{q-2}\bar{g}_3 g = |g|_k^{q-2}|g|_k^2 = |g|_k^q \tag{††}$$

can be written by the known equality $W\overline{W}_3 = |W|_k^2$. Besides, we get

$$|f|_k = |g|_k^{q-2}|\bar{g}_3|_k = |g|_k^{q-1}$$

by (†) and

$$|f|_k^p = |g|_k^{p(q-1)} = |g|_k^q \tag{*}$$

can be obtained since $p(q-1) = q$. Then we have

$$F(f) = \int_\Omega fg d\mu = \int_\Omega |g|_k^q d\mu = \|g\|_{q,\mathbb{D}}^q, \tag{1.8}$$

by (††) and then

$$\int_\Omega |g|_k^q d\mu = \|g\|_{q,k}^q = \|g\|_{q,\mathbb{D}}^q = \|g\|_{q,\mathbb{D}}^{p(q-1)} = \frac{\|g\|_{q,\mathbb{D}}^{pq}}{\|g\|_{q,\mathbb{D}}^p},$$

where

$$\|g\|_{q,\mathbb{D}}^p \int_\Omega |g|_k^q d\mu = \|g\|_{q,\mathbb{D}}^p \|g\|_{q,k}^q = \|g\|_{q,\mathbb{D}}^{pq}.$$

(**)

Therefore, by using (*) and (**)

$$\|g\|_{q,\mathbb{D}}^{p}\int_{\Omega}|g|_{k}^{q}d\mu=\|g\|_{q,\mathbb{D}}^{p}\int_{\Omega}|f|_{k}^{p}d\mu=\|g\|_{q,\mathbb{D}}^{pq}$$

can be written. From here, we get

$$\|g\|_{q,\mathbb{D}}^{q}=\|f\|_{p,\mathbb{D}}\|g\|_{q,\mathbb{D}}.$$

As a result, by (1.8)

$$|F(f)|_{k}\succcurlyeq F(f)=\|g\|_{q,\mathbb{D}}\|f\|_{p,\mathbb{D}},$$

and

$$\frac{|F(f)|_{k}}{\|f\|_{p,\mathbb{D}}}\succcurlyeq\|g\|_{q,\mathbb{D}}$$

can be obtained. Therefore, there is a function $f=|g|_{k}^{q-2}\bar{g}_{3}$ satisfying

$$\|F\|\succcurlyeq\|g\|_{q,\mathbb{D}}.$$

Consequently, the norm attains the supremum and $\|F\|=\|g\|_{q,\mathbb{D}}$ by (1.7).

Lemma 1. Let $(\Omega,\mathcal{M},\mu)$ be a finite measure space. Let $g\in L_{\mathbb{BC}}^{1}(\Omega,\mathcal{M},\mu)$ be such that for any $M\succ 0$ and for every simple function $s\in\mathbb{S}=\{s=s_{1}e_{1}+s_{2}e_{2}\mid s_{1},\ s_{2}\in S\}$ the following inequality

$$\left|\int_{\Omega}sgd\mu\right|_{k}\preccurlyeq M\|s\|_{p,\mathbb{D}}$$

holds for all $1\leq p<\infty$. Then $g\in L_{\mathbb{BC}}^{q}(\Omega,\mathcal{M},\mu)$ with $\|g\|_{q,\mathbb{D}}\preccurlyeq M$, where p and q are the conjugates.

Proof. The proof will be seperated in two cases.

Case I. Let $p = 1$ and $B = \{x \in \Omega: M \prec |g(x)|_k\}$. It is easy to see that B is in $\mathcal{M}$. If we choose the function $s = e_1\chi_B + e_2\chi_B$, then by hypothesis we have

$$\left|\int_\Omega sgd\mu\right|_k = \left|\int_\Omega (e_1\chi_B + e_2\chi_B)gd\mu\right|_k = \left|\int_B gd\mu\right|_k$$
$$\preccurlyeq \int_B |g|_k d\mu \preccurlyeq M\|\chi_B\|_{1,\mathbb{D}}$$

namely

$$\left|\int_B gd\mu\right|_k \preccurlyeq \int_B |g|_k d\mu \preccurlyeq M\|\chi_B\|_{1,\mathbb{D}} = M\big(e_1\mu_1(B) + e_2\mu_2(B)\big) = \int_B M d\mu.$$

Then one can see that

$$\int_B (|g|_k - M)d\mu \preccurlyeq 0.$$

Since $|g|_k \succ M$, we can conclude that $\mu(B) = 0$ which means that $|g(x)|_k \preccurlyeq M$ (μ-a.e.) and so $\|g\|_{\infty,\mathbb{D}} \preccurlyeq M$. As a result, the lemma is proved for Case I.

Case II. Let $1 < p < \infty$. Since $|g|_k^q \succ 0$, there exists a sequence of nonnegative simple functions $\{s_n\}_{n\in\mathbb{N}}$ such that $s_n \overset{\mathbb{D}}{\to} |g|_k^q$ pointwise by Proposition 5. Let $t_n = s_n^{\frac{1}{p}} \cdot \mathrm{sgn}_{\mathbb{D}}(g)$ be a sequence derived from $\{s_n\}_{n\in\mathbb{N}}$ where $\mathrm{sgn}_{\mathbb{D}}(g) = \bar{g}_3/|g_3|_k$ and $\bar{g}_3$ is the third conjugate of g. Note that each t_n is a simple function and

$$\|t_n\|_{p,\mathbb{D}} = \left(\int_\Omega |t_n|_k^p d\mu\right)^{\frac{1}{p}} = \left(\int_\Omega \left|s_n^{\frac{1}{p}} \cdot \mathrm{sgn}_{\mathbb{D}}(g)\right|_k^p d\mu\right)^{\frac{1}{p}} = \left(\int_\Omega |s_n|_k d\mu\right)^{\frac{1}{p}}$$
$$= \|s_n\|_{1,\mathbb{D}}^{\frac{1}{p}}.$$

Since

$$gt_n = gs_n^{\frac{1}{p}} \cdot \mathrm{sgn}_{\mathbb{D}}(g) = s_n^{\frac{1}{p}} \cdot |g|_k \succcurlyeq s_n^{\frac{1}{p}} s_n^{\frac{1}{q}} = s_n,$$

it can be written by the hypothesis that

$0 \preccurlyeq \int_\Omega s_n d\mu \preccurlyeq \int_\Omega g \cdot t_n d\mu \preccurlyeq M \|t_n\|_{p,\mathbb{D}}$.

Therefore, we get

$$\int_\Omega s_n d\mu \preccurlyeq M^q.$$

By using the bicomplex monotone convergence theorem [Theorem 3.7] in (Ghosh & Mondal, 2022), we can conclude that

$$\int_\Omega |g|_k^q d\mu \preccurlyeq M^q,$$

where $g \in L_{\mathbb{BC}}^q(\Omega, \mathcal{M}, \mu)$ and $\|g\|_{q,\mathbb{D}} \preccurlyeq M$.

Theorem 5. (*Riesz Representation Theorem for* $\mathbb{BC}$-*Lebesgue spaces*). Let $(\Omega, \mathcal{M}, \mu)$ be a σ-finite measure space and $1 \leq p < \infty$. If T is a linear functional in $L_{\mathbb{BC}}^p(\mu)$, then there exists a unique function g in $L_{\mathbb{BC}}^q(\mu)$ such that

$$T(f) = \int_\Omega fg d\mu \tag{1.9}$$

for all elements of $L_{\mathbb{BC}}^p(\mu)$ and

$$\|T\| = \|g\|_{q,\mathbb{D}} \tag{1.10}$$

where $p,\ q$ are the conjugates.

Proof. At first, the uniqueness of g will be shown. For this, suppose that there exists functions $g_1,\ g_2 \in L_{\mathbb{BC}}^q(\mu)$ such that satisfy (1.9), namely

$$\int_E g_1\, d\mu = \int_E g_2\, d\mu$$

for all $E \in \mathcal{M}$ with $\mu(E) \prec \infty_{\mathbb{D}}$ where

$$g_1(x) = e_1 g_1^{(1)}(x) + e_2 g_1^{(2)}(x) \quad \text{and} \quad g_2(x) = e_1 g_2^{(1)}(x) + e_2 g_2^{(2)}(x).$$

Since $(\Omega, \mathcal{M}, \mu)$ is a σ-finite measure space, we can find a sequence of disjoints sets $\{\Omega_n\}_{n\in\mathbb{N}}$ in $\mathcal{M}$ such that $\mu(\Omega_n) = e_1\mu_1(\Omega_n) + e_2\mu_2(\Omega_n) \prec \infty_{\mathbb{D}}$ for all n and

$$\Omega = \bigcup_{n=1}^{\infty} \Omega_n.$$

Now, let $A := \{x \in \Omega : g_1(x) \succ g_2(x)\}$ and $B := \{x \in \Omega : g_1(x) \prec g_2(x)\}$. Then

$$\int_{\Omega_n \cap A} g_1 d\mu = \int_{\Omega_n \cap A} g_2 d\mu$$

and so

$$0 = \int_{\Omega_n \cap A} (g_1 - g_2) d\mu$$

$$= e_1 \int_{\Omega_n \cap A} \left(g_1^{(1)} - g_2^{(1)}\right) d\mu_1$$

$$+ e_2 \int_{\Omega_n \cap A} \left(g_1^{(2)} - g_2^{(2)}\right) d\mu_2.$$

Since $g_1(x) \succ g_2(x)$ i.e. $g_1^{(1)}(x) > g_2^{(1)}(x)$ and $g_1^{(2)}(x) > g_2^{(2)}(x)$ for all $x \in \Omega_n \cap A$, we can find that $\mu_1(\Omega_n \cap A) = 0$ and $\mu_2(\Omega_n \cap A) = 0$ for all $n \in \mathbb{N}$. Then

$$\mu(A) = e_1\mu_1(A) + e_2\mu_2(A)$$
$$= e_1 \sum_{n=1}^{\infty} \mu_1 (A \cap \Omega_n) + e_2 \sum_{n=1}^{\infty} \mu_2 (A \cap \Omega_n) = 0.$$

Similarly $\mu(B) = 0$ and $g_1^{(1)} = g_2^{(1)}$ (μ_1 -a.e), $g_1^{(2)} = g_2^{(2)}$ (μ_2 -a.e). Therefore $g_1 = g_2$ (μ-a.e.) and this proves the uniqueness.

Now we'll prove the existence of g by cases.

Case 1. Let $\mu(\Omega) \prec \infty_{\mathbb{D}}$ and define

$$e_1 v_1(E) + e_2 v_2(E) = v(E) = T(\chi_E) = e_1 T_1(\chi_E) + e_2 T_2(\chi_E)$$

for each $E \in \mathcal{M}$. Since $\mu(\Omega) \prec \infty_{\mathbb{D}}$, we get $\mu_1(E),\ \mu_2(E) < \infty$. Thus

$$\begin{aligned}\|\chi_E\|_{p,\mathbb{D}} &= e_1\|\chi_E\|_{p,\mu_1} + e_2\|\chi_E\|_{p,\mu_2} \\ &= e_1\mu_1(E)^{\frac{1}{p}} + e_2\mu_2(E)^{\frac{1}{p}} = \mu(E)^{\frac{1}{p}}\end{aligned}$$

by Definition 2.2 in (Değirmen & Sağır, 2023). This means $\chi_E \in L^p_{\mathbb{BC}}(\mu)$. Now, it will be shown that $v = e_1 v_1 + e_2 v_2$ is a $\mathbb{D}$-signed measure on $\mathcal{M}$. It is easy to see that $\chi_\emptyset$ is the zero function in $L^p_{\mathbb{BC}}(\mu)$ and so $v(\emptyset) = T(\chi_\emptyset)$. Since T is a bicomplex function, $v(\cdot)$ is also a bicomplex function. Likewise, let $\{E_n\}_{n\in\mathbb{N}}$ be a collection of disjoint sets in $\mathcal{M}$ and define

$$E = \bigcup_{n=1}^{\infty} E_n \quad \text{and} \quad A_n = \bigcup_{i=1}^{n} E_i.$$

Then the sequence $\{A_n\}_{n\in\mathbb{N}}$ is increasing and

$$\bigcup_{n=1}^{\infty} A_n = E.$$

Therefore, one can write

$$\chi_{A_n} = \sum_{k=1}^{n} \chi_{E_k}$$

by induction and

$$\begin{aligned}T(\chi_{A_n}) &= e_1T_1(\chi_{A_n}) + e_1T_2(\chi_{A_n}) = e_1\sum_{k=1}^{n} T_1\left(\chi_{E_k}\right) + e_2\sum_{k=1}^{n} T_2\left(\chi_{E_k}\right) \\ &= e_1\sum_{k=1}^{n} v_1\left(E_k\right) + e_2\sum_{k=1}^{n} v_2\left(E_k\right) = \sum_{k=1}^{n} v\left(E_k\right)\end{aligned}$$

by the linearity of T. Since

$$\left\|\chi_{A_n}-\chi_E\right\|_{p,\mathbb{D}} = e_1\left\|\chi_{A_n}-\chi_E\right\|_{p,\mu_1} + e_2\left\|\chi_{A_n}-\chi_E\right\|_{p,\mu_2}$$
$$= e_1\left(\int_\Omega \left|\chi_{A_n}-\chi_E\right|^p d\mu_1\right)^{\frac{1}{p}} + e_2\left(\int_\Omega \left|\chi_{A_n}-\chi_E\right|^p d\mu_2\right)^{\frac{1}{p}}$$
$$= e_1\mu_1(E\backslash A_n)^{\frac{1}{p}} + e_1\mu_2(E\backslash A_n)^{\frac{1}{p}} = \left(\mu(E)-\mu(A_n)\right)^{\frac{1}{p}}$$

and $\{A_n\}_{n\in\mathbb{N}}$ is an increasing sequence, we have $\mu(E) = \lim_{n\to\infty}{}_{\mathbb{D}}\mu(A_n)$ namely

$$\lim_{n\to\infty}{}_{\mathbb{D}}\left\|\chi_{A_n}-\chi_E\right\|_{p,\mathbb{D}} = 0.$$

Using the $\mathbb{D}$ −continuity of T in $L^p_{\mathbb{BC}}(\mu)$, it follows that

$$\lim_{n\to\infty}{}_{\mathbb{D}}T\left(\chi_{A_n}\right) = T(\chi_E)$$

and

$$v(E) = e_1v_1(E) + e_2v_2(E) = e_1T_1(\chi_E) + e_2T_2(\chi_E)$$
$$= e_1\lim_{n\to\infty}T_1\left(\chi_{A_n}\right) + e_2\lim_{n\to\infty}T_2\left(\chi_{A_n}\right) = lim_{\mathbb{D}\,n\to\infty}T\left(\chi_{A_n}\right) = lim_{\mathbb{D}\,n\to\infty}\sum_{k=1}^{n} v\,(E_k).$$

This last equality says that v is a $\mathbb{D}$-signed measure. Now, $\mathbb{D}$ −absolute continuity of v with respect to μ will be proved ($v \ll_{\mathbb{D}} \mu$) by using Theorem 3.12 in (Ghosh & Mondal, 2022). Suppose that $E \in \mathcal{M}$ with $\mu(E) = 0$. Then

$$\|\chi_E\|_{p,\mathbb{D}} = e_1\|\chi_E\|_{p,\mu_1} + e_2\|\chi_E\|_{p,\mu_2}$$
$$= e_1\mu_1(E)^{\frac{1}{p}} + e_2\mu_2(E)^{\frac{1}{p}} = \mu(E)^{\frac{1}{p}}.$$

This says that χ_E is equal to the zero function in $L^p_{\mathbb{BC}}(\Omega,\mathcal{M},\mu)$ and $T(\chi_E) = 0$, namely $v(E) = 0$ and so $v \ll_{\mathbb{D}} \mu$. By Theorem 3.16 of (Ghosh & Mondal, 2022), the bicomplex version of Lebesgue-Radon-Nikodym theorem for measures (signed) finite, there is a bicomplex measurable function g such that

$$v(E) = \int_E g d\mu$$

for all $E \in \mathcal{M}$. Therefore

$$\int_\Omega g d\mu = \int_\Omega (e_1 g_1 + e_2 g_2)(e_1 d\mu_1 + e_2 d\mu_2)$$
$$= e_1 \int_\Omega g_1 d\mu_1 + e_2 \int_\Omega g_2 d\mu_2 = e_1 v_1(\Omega) + e_2 v_2(\Omega)$$
$$= v(\Omega) = T(\chi_\Omega) = T(1) \prec \infty_{\mathbb{D}},$$

and $g \in L^1_{\mathbb{BC}}(\Omega, \mathcal{M}, \mu)$. Let us check whether g meets the hypotheses of Lemma 1. Let $s \in L^p_{\mathbb{BC}}(\Omega, \mathcal{M}, \mu)$ be a simple $\mathcal{M}$-measurable function with the following canonical representation

$$s = e_1 s_1 + e_2 s_2 = e_1 \sum_{k=1}^n \alpha_k^{(1)} \chi_{E_k} + e_2 \sum_{k=1}^n \alpha_k^{(2)} \chi_{E_k}.$$

Then, by using the definition of T, we get

$$T(s) = e_1 T_1(s_1) + e_2 T_2(s_2) = e_1 T_1\left(\sum_{k=1}^n \alpha_k^{(1)} \chi_{E_k}\right) + e_2 T_2\left(\sum_{k=1}^n \alpha_k^{(2)} \chi_{E_k}\right)$$
$$= e_1 \sum_{k=1}^n \alpha_k^{(1)} v_1(E_k) + e_2 \sum_{k=1}^n \alpha_k^{(2)} v_2(E_k) = e_1 \sum_{k=1}^n \alpha_k^{(1)} \int_{E_k} g_1 d\mu_1 + e_2 \sum_{k=1}^n \alpha_k^{(2)} \int_{E_k} g_2 d\mu_2$$
$$= e_1 \int_\Omega g_1 \left(\sum_{k=1}^n \alpha_k^{(1)} \chi_{E_k}\right) d\mu_1 + e_2 \int_\Omega g_2 \left(\sum_{k=1}^n \alpha_k^{(2)} \chi_{E_k}\right) d\mu_2$$
$$= e_1 \int_\Omega g_1 s_1 d\mu_1 + e_2 \int_\Omega g_2 s_2 d\mu_2 = \int_\Omega g s d\mu .$$

Therefore, we have

$$T(s) = \int_\Omega g s d\mu$$

for all (step) simple function $s \in L^p_{\mathbb{BC}}(\Omega, \mathcal{M}, \mu)$. As a result of this, $|T(s)|_k = \left|\int_\Omega g s d\mu\right|_k$.

If $M = \|T\|$, then M is $\mathbb{D}-$finite and it demonstrates that g meets the criteria of Lemma 1. Therefore we can conclude that $g \in L^q_{\mathbb{BC}}(\Omega, \mathcal{M}, \mu)$ and

$$\|g\|_{q,\mathbb{D}} \preccurlyeq \|T\| = M. \tag{1.11}$$

Now, we shall demonstrate that $T(f) = \int_\Omega gfd\mu$ for any $f \in L^p_{\mathbb{BC}}(\Omega, \mathcal{M}, \mu)$. Let $f \in L^p_{\mathbb{BC}}(\Omega, \mathcal{M}, \mu)$ and $\varepsilon = e_1\varepsilon_1 + e_2\varepsilon_2 \succ 0$. Since the set, obtained from simple functions, $\mathbb{S} = \{s = s_1e_1 + s_2e_2 \mid s_1,\ s_2 \in S\}$ is $\mathbb{D}-$dense in $L^p_{\mathbb{BC}}(\Omega, \mathcal{M}, \mu)$, one can find a simple function $s = e_1s_1 + e_2s_2 \in \mathbb{S} \subset L^p_{\mathbb{BC}}(\Omega, \mathcal{M}, \mu)$ such that

$$\|f - s\|_{p,\mathbb{D}} \prec \frac{\varepsilon}{\|g\|_{q,\mathbb{D}} + \|T\| + 1}.$$

Then

$$\left|T(f) - \int_\Omega gfd\mu\right|_k = \left|T(f) - T(s) + T(s) - \int_\Omega gfd\mu\right|_k \preccurlyeq |T(f) - T(s)|_k + \left|\int_\Omega sgd\mu - \int_\Omega gfd\mu\right|_k$$
$$= |T(f - s)|_k + \left|\int_\Omega sgd\mu - \int_\Omega gfd\mu\right|_k \preccurlyeq |T(f - s)|_k + \int_\Omega |g|_k |s - f|_k d\mu$$
$$\preccurlyeq \|T\|\|f - s\|_{p,\mathbb{D}} + \|g\|_{q,\mathbb{D}}\|f - s\|_{p,\mathbb{D}} = \|f - s\|_{p,\mathbb{D}}\left(\|T\| + \|g\|_{q,\mathbb{D}}\right)$$
$$\preccurlyeq \|f - s\|_{p,\mathbb{D}}\left(\|T\| + \|g\|_{q,\mathbb{D}} + 1\right) \prec \varepsilon.$$

Since ε is arbitrary, we can conclude that

$$T(f) = \int_\Omega gfd\mu$$

for all $f \in L^p_{\mathbb{BC}}(\Omega, \mathcal{M}, \mu)$. Finally by using the Hölder's inequality (Theorem 1), we can write $|T(f)|_k \preccurlyeq \|f\|_{p,\mathbb{D}}\|g\|_{q,\mathbb{D}}$ where

$$\|T\| \preccurlyeq \|g\|_{q,\mathbb{D}}. \tag{1.12}$$

Now from (1.11) and (1.12) we have $\|T\| = \|g\|_{q,\mathbb{D}}$, which is done for the Case 1.

Case 2. $\mu(\Omega) = \infty$. Under the σ-finiteness of μ, there exists a collection of measurable sets $\{\Omega_n\}_{n\in\mathbb{N}}$ such that $\Omega = \bigcup_{n=1}^\infty \Omega_n$, $\Omega_n \subset \Omega_{n+1}$ and $\mu(\Omega_n) \prec \infty_\mathbb{D}$ for all $n \in \mathbb{N}$. Therefore, Case 1 may be applied to the measure space $(\Omega_n, \mathcal{M} \cap \Omega_n, \mu_n)$ where μ_n is the restriction of μ to $\mathcal{M} \cap$

Ω_n. Now let $T_n = T|_{L^p_{\mathbb{BC}}(\mu_n)}$. Then, there exists $g_n \in L^q_{\mathbb{BC}}(\mu_n)$ for all $n \in \mathbb{N}$ such that

$$T_n(h) = \int_{\Omega_n} hg_n d\mu_n \tag{1.13}$$

for all $h \in L^p_{\mathbb{BC}}(\Omega, \mathcal{M}, \mu)$ which vanishes outside Ω_n by Case 1. Also

$$\|T\| \succcurlyeq \|T_n\| = \|g_n\|_{q,\mathbb{D}}. \tag{1.14}$$

If we define

$$\tilde{g}_n(x) = \begin{cases} g_n(x), & x \in \Omega_n \\ 0, & x \notin \Omega_n \end{cases}$$

then, the integral in (1.13) can be written as

$$T_n(h) = \int_{\Omega} h\tilde{g}_n d\mu_n \tag{1.15}$$

for all $h \in L^p_{\mathbb{BC}}(\Omega, \mathcal{M}, \mu)$ which vanishes outside Ω_n. Since $\tilde{g}_{n+1}$ restricted to Ω_n have the same properties as $\tilde{g}_n$ under the uniqueness, we have $\tilde{g}_{n+1} = \tilde{g}_n$ in Ω_n. Now define $g(x) = g_n(x)$ if $x \in \Omega_n$. Since $|\tilde{g}_n(x)|_k \preccurlyeq |\tilde{g}_{n+1}(x)|_k$ for all $x \in \Omega$ and

$$\lim_{n\to\infty}{}_{\mathbb{D}} \tilde{g}_n(x) = g(x),$$

we can say that

$$\int_{\Omega} |g|_k^q d\mu = \lim_{n\to\infty}{}_{\mathbb{D}} \int_{\Omega} |\tilde{g}_n|_k^q d\mu \preccurlyeq \|T\|^q$$

by the bicomplex monotone convergence theorem in (Ghosh & Mondal, 2022). This implies that $g \in L^q_{\mathbb{BC}}(\Omega, \mathfrak{M}, \mu)$ and

$$\|T\| \succcurlyeq \|g\|_{q,\mathbb{D}}. \tag{1.16}$$

Let f be any element of $L^p_{\mathbb{BC}}(\Omega, \mathcal{M}, \mu)$ and $f_n = f \cdot \chi_{\Omega_n}$. Then f_n vanishes outside Ω_n and the pointwise convergence $f_n \overset{\mathbb{D}}{\to} f$ exists in Ω. It is easy to see that $|f_n - f|_k \preccurlyeq ||f|_k$ and so $|f_n - f|_k^p \preccurlyeq ||f|_k^p$. By the bicomplex Dominated convergence theorem, we have

$$\lim_{n\to\infty}{}_{\mathbb{D}}\int_{\Omega}|f_n - f|_k^p d\mu = 0.$$

The $\mathbb{D}-$ continuity of T implies that $T(f_n)\overset{\mathbb{D}}{\to}T(f)$ when $n\to\infty$. Moreover, we have $|f_n\cdot g|_k = |f_n|_k|g|_k \preccurlyeq |f|_k|g|_k = |f\cdot g|_k$, $f\cdot g\in L^1_{\mathbb{BC}}(\Omega,\mathcal{M},\mu)$ and $\lim_{n\to\infty}{}_{\mathbb{D}} f_n g = fg$. If we apply the bicomplex Dominated convergence theorem once more, we get

$$\begin{aligned}\int_{\Omega} fg d\mu &= \lim_{n\to\infty}{}_{\mathbb{D}}\int_{\Omega} f_n g d\mu = \lim_{n\to\infty}{}_{\mathbb{D}}\int_{\Omega} f\chi_{\Omega_n} g d\mu \\ &= \lim_{n\to\infty}{}_{\mathbb{D}}\int_{\Omega}\left(f\chi_{\Omega_n}\right)\left(g\chi_{\Omega_n}\right)d\mu = \lim_{n\to\infty}{}_{\mathbb{D}}\int_{\Omega}(f_n)\tilde{g}_n d\mu \\ &= \lim_{n\to\infty}{}_{\mathbb{D}} T(f_n) = T(f).\end{aligned}$$

Thus, we get (1.9) and applying the Hölder's inequality once more,

$$|T(f)|_k \preccurlyeq \|f\|_{p,\mathbb{D}}\|g\|_{q,\mathbb{D}}$$

can be written. This means $\|T\| \preccurlyeq \|g\|_{q,\mathbb{D}}$, and so $\|T\| = \|g\|_{q,\mathbb{D}}$ by (1.16), which ends the proof.

Corollary 1. The space $L^p_{\mathbb{BC}}(\Omega,\mathcal{M},\mu)$ with $1<p<\infty$ is reflexive.

REFERENCES

Alpay, D., Luna-Elizarrarás, M.E., Shapiro, M. & Struppa,D.C. (2014) *Basics of functional analysis with bicomplex scalars, and bicomplex Schur analysis*. Springer Science & Business Media. Doi: 10.1007/978-3-319-05110-9

Castillo, R.E. & Rafeiro, H. (2016) *An Introductory Course in Lebesgue Spaces*. Springer International Publishing Switzerland. Doi: 10.1007/978-3-319-30034-4

Colombo, F., Sabatini, I. &Struppa, D. (2014) Bicomplex holomorphic functional calculus. *Math. Nachr.*, *287*(10), 1093-1105. Doi:10.1002/mana.201200354

Değirmen, N. &Sağır, B. (2023) On bicomplex $\mathbb{BC}$ modules $l_p^k(\mathbb{BC})$ and some of their geometric properties. *Georgian Mathematical Journal*, *30* (1), 65-79. Doi: 10.1515/gmj-2022-2194

Dubey, S., Kumar, R. & Sharma,K. (2014) A note on bicomplex Orlicz spaces. arXiv:1401.7112, 1-12.

Ghosh, C. & Mondal, S. (2022) Bicomplex Version of Lebesgue's Dominated Convergence Theorem and Hyperbolic Invariant Measure. *Adv. in Appl. Clifford Algebras, 32*(3), 32-37. Doi: 10.1007/s00006-022-01216-0

Koumantos, P.N. (2023) On the Mean Ergodic Theorem in Bicomplex Banach Modules. *Adv. in Appl. Clifford Algebras, 33*(1), 1-14. Doi: 10.1007/s00006-023-01262-2

Kumar, R. & Saini, H. (2016) Topological bicomplex modules, *Adv. Appl. in Clifford Algebras, 26*, 1249-1270. Doi: 10.1007/s00006-016-0646-1

Kumar, R., Kumar, R. & Rochon, D. (2011) The fundamental theorems in the framework of bicomplex topological modules, arXiv:1109.3424v1.

Luna-Elizarrarás, M.E., Perez-Regalado, C.O. & Shapiro, M. (2014) On linear functionals and Hahn–Banach theorems for hyperbolic and bicomplex modules, *Adv. Appl. in Clifford Algebras, 24*, 1105-1129. Doi: 10.1007/s00006-014-0503-z

Luna-Elizarrarás, M.E., Shapiro, M., Struppa, D.C. & Vajiac, A. (2015) *Bicomplex holomorphic functions: the algebra, geometry and analysis of bicomplex numbers*, Birkhäuser. Doi: 10.1007/978-3-319-24868-4

Price, G.B. (2018) *An introduction to multicomplex spaces and functions*, CRC Press.

Sağır, B., Değirmen, N. & Duyar, C. (2023) Bicomplex matrix transformations between c_0 and c in bicomplex setting. *U.P.B. Sci. Bull., Series A, 85* (4), 115-122.

Saini, H., Sharma, A. & Kumar, R. (2020) Some fundamental theorems of functional analysis with bicomplex and hyperbolic scalars, *Adv. in Appl. Clifford Algebras, 30*, 1-23. Doi: 10.1007/s00006-020-01092-6

Toksoy, E. & Sağır, B. (2023) On geometrical characteristics and inequalities of new bicomplex Lebesgue Spaces with hyperbolic-valued norm. *Georgian Mathematical Journal*, Doi: 10.1515/gmj-2023-2093

CHAPTER IV

On Geometric Structure of Bicomplex Weighted $\mathbb{BC}$ −Modules $l_p^k(\mathbb{BC})$

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Introduction

In recent years, a novel number system known as bicomplex numbers has emerged as an expansion of the existing system of complex numbers. The book written by Price (Price, 1991) is the foremost recommended beginning work for bicomplex numbers. Next, we recommend books (Alpay & et al., 2014) and (Luna-Elizarraras & et al., 2015) that provide detailed explanations of the

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fundamental frameworks of bicomplex numbers, which concentrate on functional analysis.

The following fundamental details concerning bicomplex numbers are derived from books (Alpay & et al., 2014; Luna-Elizarraras & al., 2015; Price, 1991). Let us begin by examining two separate imaginary numbers, i and j, where $i^2 = j^2 = -1$. This results in the creation of two sets, $\mathbb{C}(i)$ and $\mathbb{C}(j)$, which are exactly the same as set $\mathbb{C}$. Additionally, let us take an imaginary number, denoted as k, which is the result of multiplying these two imaginary numbers and possesses the following characteristics:

$$ij = ji = k$$

$$ik = ki = -j$$

$$jk = kj = -i$$

$$k^2 = 1.$$

Consequently, the set of bicomplex numbers denoted by $\mathbb{BC}$ is given as

$$\mathbb{BC} = \{z + jw | z, w \in \mathbb{C}(i)\}.$$

The set $\mathbb{BC}$ can also be written as

$$\mathbb{BC} = \{a + bi + cj + dk | a, b, c, d \in \mathbb{R}\}.$$

Addition and multiplication operations are defined in the following manners:

$$(z_1 + jw_1) + (z_2 + jw_2) = (z_1 + z_2) + j(w_1 + w_2)$$

and

$$(z_1 + jw_1)(z_2 + jw_2) = (z_1w_1 - w_1w_2) + j(z_1w_2 + w_1z_2).$$

By undertaking these operations, the set $\mathbb{BC}$ improves into a commutative ring and, as a result, becomes a module on itself. A valuable subset of $\mathbb{BC}$ is denoted by $\mathbb{D}$, which is called the hyperbolic numbers set, and is properly described as

$$\mathbb{D} = \{x + ky | x, y \in \mathbb{R}\}$$

by replacing z and w with x and iy in the description of $\mathbb{BC}$, whenever $x, y \in \mathbb{R}$. Now, let us examine two complex numbers,

$$e_1 = \frac{1+ij}{2} \text{ and } e_2 = \frac{1-ij}{2}.$$

These particular numbers possess the following properties, which are readily apparent:

$$e_1 e_2 = 0, e_1 + e_2 = 1, e_1 e_1 = e_1, e_2 e_2 = e_2.$$

Both the numbers e_1 and e_2 belong to the set of hyperbolic numbers $\mathbb{D}$. They constitute the idempotent basis of bicomplex numbers. A bicomplex number $z + jw$ can be expressed in a unique form:

$$z + jw = e_1 z_1 + e_2 z_2,$$

where $z_1 = z - iw$, $z_2 = z + iw \in \mathbb{C}(i)$. The given mathematical expression indicates the idempotent representation of bicomplex numbers. Therefore, it ensures that every hyperbolic number possesses an idempotent representation, that can be expressed as:

$$x + ky = e_1 a_1 + e_2 a_2,$$

where $a_1 = x + y$, $a_2 = x - y \in \mathbb{R}$. The non-negative hyperbolic numbers are represented as $\mathbb{D}^+$ and given by

$$\mathbb{D}^+ = \{e_1 a_1 + e_2 a_2 | a_1, a_2 \geq 0\}.$$

Consider $\kappa, \mu \in \mathbb{D}$. The hyperbolic numbers are equipped with a partial order relation denoted by $\lesssim$, which is given as:

$$\kappa \lesssim \mu \text{ if and only if } \mu - \kappa \in \mathbb{D}^+.$$

Moreover, we write

$$\kappa \prec \mu \text{ if and only if } \kappa \lesssim \mu \text{ but } \kappa \neq \mu.$$

To find out more concerning the features of the partial order relation $\lesssim$, please consult Section 2.6.2 of (Luna-Elizarraras & et al., 2015). Let us now provide a brief overview of $\mathbb{D}$-boundedness as presented in Section 2.6.3 of (Luna-Elizarraras & et al., 2015). Let $X \subset \mathbb{D}$. Assuming X possesses a $\mathbb{D}$-upper bound (or a $\mathbb{D}$-lower bound) l, it implies that for any $x \in X$, x can be compared to l and $x \precsim l$ (or $l \precsim x$). The set X can be described as $\mathbb{D}$-bounded if it is $\mathbb{D}$-bounded both from above and from below. If X is a set that has a $\mathbb{D}$-upper bound, we determine the concept of its $\mathbb{D}$-supremum, represented as

$$\sup_{\mathbb{D}} X,$$

as the least $\mathbb{D}$-upper bound for X. Similarly, we establish its $\mathbb{D}$-infimum

$$\inf_{\mathbb{D}} X,$$

as the greatest $\mathbb{D}$-lower bound for X. In this context, the least $\mathbb{D}$-upper bound denotes

$$\sup_{\mathbb{D}} X \precsim l$$

for any $\mathbb{D}$-upper bound l, even though not all $\mathbb{D}$-upper bounds are comparable to one another. Likewise, the greatest $\mathbb{D}$-lower bound signifies that

$$l \precsim \inf_{\mathbb{D}} X$$

for any $\mathbb{D}$-lower bound l, even though not all $\mathbb{D}$-lower bounds are comparable to one another. Consider the sets

$$X_1 = \{x_1 : x_1 e_1 + x_2 e_2 \in X\}$$

and

$$X_2 = \{x_2 : x_1 e_1 + x_2 e_2 \in X\}.$$

If X is $\mathbb{D}$-bounded from above, then

$$\sup_{\mathbb{D}} X = \sup X_1 e_1 + \sup X_2 e_2.$$

If X is $\mathbb{D}$-bounded from below, then

$$\inf_{\mathbb{D}} X = \inf X_1 e_1 + \inf X_2 e_2.$$

Additionally, the following items satisfy:

(i) If both X and Y have a $\mathbb{D}$-lower bound, then $X+Y$ also has a $\mathbb{D}$-lower bound and

$$\inf_{\mathbb{D}}(X+Y)=\inf_{\mathbb{D}}X+\inf_{\mathbb{D}}Y.$$

(ii) If both X and Y have a $\mathbb{D}$-upper bound, then $X+Y$ also has a $\mathbb{D}$-upper bound and

$$\sup_{\mathbb{D}}(X+Y)=\sup_{\mathbb{D}}X+\sup_{\mathbb{D}}Y.$$

For a bicomplex number $z+jw$, $|\mathrm{z}+\mathrm{jw}|_k$ represents the hyperbolic modulus and is indicated by

$$|z+jw|_k^2=(z+jw)(\overline{z}-j\overline{w}).$$

By representing the complex number $z+jw$ in idempotent form, we obtain

$$|z+jw|_k=|e_1z_1+e_2z_2|_k=e_1|z_1|+e_2|z_2|. \qquad (1)$$

Modules are defined as algebraic structures that are constructed over rings, similar to how vector spaces are constructed over fields. A module X is referred to as a $\mathbb{BC}$-module if it is defined over the ring $\mathbb{BC}$ (see (Gervais Lavoie & et al., 2011; Rochon & Tremblay, 2006)). Conside E be a $\mathbb{BC}$-module. Let us take $u,v\in E$ and $\lambda\in\mathbb{BC}$. A function $\|.\|_{\mathbb{D}}: E\to\mathbb{D}^+$ is considered to be a hyperbolic (D-valued) norm on E if it satisfies the conditions listed below:

(i) $\|u\|_{\mathbb{D}}=0$ if and only if $u=0$.

(ii) $\|\lambda u\|_{\mathbb{D}}=|\lambda|_k\|u\|_{\mathbb{D}}$.

(iii) $\|u+v\|_{\mathbb{D}}\lesssim\|u\|_{\mathbb{D}}+\|v\|_{\mathbb{D}}$.

The hyperbolic modulus $|.|_k$ with representation (1) refers to the hyperbolic ($\mathbb{D}$-valued) norm of $z+jw\in\mathbb{BC}$. Let $\upsilon,\eta\in\mathbb{BC}$ and $\gamma\in\mathbb{D}^+$. The following characteristics are thereby associated with the hyperbolic ($\mathbb{D}$ -valued) norm $|.|_k$:

(i) $|\upsilon|_k=0$ if and only if $\upsilon=0$.

(ii) $|\upsilon\eta|_k=|\upsilon|_k|\eta|_k$.

(iii) $|\upsilon+\eta|_k \precsim |\upsilon|_k + |\eta|_k$.

(iv) $|\gamma|_k = \gamma$

(v) $|\gamma\upsilon|_k = \gamma|\upsilon|_k$

Consider X as a $\mathbb{BC}$-module. Therefore, we possess

$$X = e_1 X_1 + e_2 X_2$$

such that $X_1 = e_1 X$ and $X_2 = e_2 X$ are complex vector spaces and also $\mathbb{BC}$-modules. The equation presented here is commonly referred to as the idempotent decomposition of X. Consequently, every x belonging to the set X can be expressed uniquely as $x = e_1 x_1 + e_2 x_2$, where $x_1 \in X_1$ and $x_2 \in X_2$ (see (Gervais Lavoie et al., 2010)). Let X_1 and X_2 be normed spaces equipped with norms $\|.\|_1$ and $\|.\|_2$, respectively. The $\mathbb{BC}$-module can be equipped with the hyperbolic ($\mathbb{D}$-valued) norm determined by

$$\|x\|_{\mathbb{D}} = e_1\|x_1\|_1 + e_2\|x_2\|_2.$$

Consider T is a $\mathbb{BC}$-module with the hyperbolic ($\mathbb{D}$-valued) norm $\|.\|_{\mathbb{D}}$. A sequence (t_n) belonging to T converges to $t_0 \in T$ concerning the hyperbolic ($\mathbb{D}$-valued) norm $\|.\|_{\mathbb{D}}$ if for every $0 \prec \epsilon$ there exists $n_0 \in \mathbb{N}$ such that $\|t_n - t_0\|_{\mathbb{D}} \prec \epsilon$ for all $n \geq n_0$. In this study, the term used for describing this convergence is $\mathbb{D}$-convergence (or $\mathbb{D}$-converges). Also, the term $\mathbb{D}$-divergent will be used for $\mathbb{D}$-non-convergent sequences.

Consider T is a $\mathbb{BC}$-module with hyperbolic ($\mathbb{D}$-valued) norm $\|.\|_{\mathbb{D}}$. A sequence (t_n) belonging to T is Cauchy sequence concerning the hyperbolic ($\mathbb{D}$-valued) norm $\|.\|_{\mathbb{D}}$ if for every $0 \prec \epsilon$ there exists $n_0 \in \mathbb{N}$ such that $\|t_n - t_m\|_{\mathbb{D}} \prec \epsilon$ for all $n, m \geq n_0$. If every Cauchy sequence in X with respect to the hyperbolic ($\mathbb{D}$-valued) norm $\|.\|_{\mathbb{D}}$ $\mathbb{D}$-converges to $t_0 \in X$, then we say that the $\mathbb{BC}$-module X is complete with respect to the hyperbolic ($\mathbb{D}$-valued) norm $\|.\|_{\mathbb{D}}$.

A bicomplex Banach module is defined as a complete real-valued normed or hyperbolic ($\mathbb{D}$-valued) normed $\mathbb{BC}$-module.

(Kumar & et al., 2011; Gervais Lavoie & et al. 2010; Gervais Lavoie & et al., 2011; Kumar & et al., 2016).

The Lebesgue sequence spaces are a family of sequence spaces that are important in functional analysis and measure theory. They are named after the French mathematician Henri Lebesgue, who made significant contributions to both fields. These spaces are defined as the collection of all sequences of complex numbers that satisfy certain convergence properties. The weighted Lebesgue sequence spaces are a class of function spaces that generalize the classical Lebesgue sequence spaces. These spaces have been extensively studied in functional analysis and have applications in various areas of mathematics and engineering. In a recent study, (Güngör, 2020) examined the geometric characteristics of non-Newtonian Lebesgue sequence spaces. Also, (Oğur, 2019), (Sağır & Alşalvar, 2019) pertain to the examination of the geometric characteristics of the weighted Lebesgue function and sequence space, respectively. Furthermore, there have been new investigations into bicomplex function and sequence Lebesgue spaces, which can be seen as extensions of Lebesgue function and sequence spaces. There is a significant amount of research in the literature that examines the geometric characteristics of bicomplex Lebesgue function spaces and Lebesgue sequence spaces. (Değirmen & Sağır, 2023) and (Toksoy & Sağır, 2023) are among the notable examples. Besides, (Değirmen & Sağır, 2021) examined the $D-$topological duals of bicomplex Lebesgue sequence spaces with hyperbolic ($\mathbb{D}$-valued) norm.

Recent work (Sağır & Güngör, 2023) introduces the weighted bicomplex sequence spaces $l_{p,\alpha}^{k}(\mathbb{BC})$ as a generalization of the bicomplex Lebesgue sequence spaces $l_{p}^{k}(\mathbb{BC})$ with hyperbolic ($\mathbb{D}$-valued) norm, where α is a bicomplex weighted sequence. Initially, this study will address the findings of earlier studies, mostly focusing on the results of studies (Sağır & Güngör, 2023) and (Değirmen & Sağır, 2023), which will serve as the foundation for the main portion of this study.

To begin, let us summarize some valuable observations gained from the findings of the study (Değirmen & Sağır, 2023).

Definition 1.1. Consider $\mathrm{w}, \lambda \in \mathbb{BC}$ and $\mathrm{w} \neq 0$. The map given by $\mathrm{w}^{\lambda} = \mathrm{e}^{\lambda \mathrm{Lnw}}$ is referred to as the principal value of the bicomplex power w^{λ}. Observe that for $\lambda = \lambda_1 \mathrm{e}_1 + \lambda_2 \mathrm{e}_2$ and w= $w_1 \mathrm{e}_1 + \mathrm{w}_2 \mathrm{e}_2$

$$\mathrm{w}^{\lambda} = \mathrm{e}^{\lambda \mathrm{Lnw}} = w_1^{\ \lambda_1} \mathrm{e}_1 + \mathrm{w}_2^{\ \lambda_2} \mathrm{e}_2$$

Particularly if $\lambda \in \mathbb{R}$, then we have

$$\mathrm{w}^{\lambda} = w_1^{\lambda} e_1 + w_2^{\lambda} e_2.$$

Lemma 1.2. Consider $\mathrm{p} \in \mathbb{R}$ with $2 \leq \mathrm{p} < \infty$. Assume that u and v are any two bicomplex numbers. Then we have

$$|\mathrm{u}+\mathrm{v}|_{\mathrm{k}}^{\mathrm{p}} + |\mathrm{u}-\mathrm{v}|_{\mathrm{k}}^{\mathrm{p}} \lesssim 2^{\mathrm{p}-1}\left(|\mathrm{u}|_{\mathrm{k}}^{\mathrm{p}} + |\mathrm{v}|_{\mathrm{k}}^{\mathrm{p}}\right).$$

Definition 1.3. Consider A is a subset of a $\mathbb{BC}$-module E. Then A is described as a $\mathbb{BC}$-convex set if $\mathrm{u}, \mathrm{v} \in \mathrm{E}$ and $\gamma \in \mathbb{D}^{+}$ satisfying $0 \lesssim \gamma \lesssim 1$ implies that $\gamma\mathrm{u} + (1-\gamma)\mathrm{v} \in \mathrm{A}$.

Let us briefly outline the significant findings derived from (Toksoy and Sağır, 2023).

Definition 1.4. Assume that B is a bicomplex Banach module with the hyperbolic ($\mathbb{D}$-valued) norm $\|.\|_{\mathrm{B},\mathbb{D}}$. Then B is described as $\mathbb{BC}$-strictly convex, if $\|\gamma\mathrm{u} + (1-\gamma)\mathrm{v}\|_{\mathrm{B},\mathbb{D}} \prec 1$ for all $0 \lesssim \gamma \lesssim 1$ with $\gamma \in \mathbb{D}^{+}$, $\mathrm{u}, \mathrm{v} \in \mathrm{B}$ with $\|\mathrm{u}\|_{\mathrm{B},\mathbb{D}} = \|\mathrm{v}\|_{\mathrm{B},\mathbb{D}} = 1$ and $\mathrm{u} \neq \mathrm{v}$.

Definition 1.5. Assume that B is a bicomplex Banach module with the hyperbolic ($\mathbb{D}$-valued) norm $\|.\|_{\mathrm{B},\mathbb{D}}$. Then B is described as $\mathbb{BC}$-uniformly convex, if for any $\epsilon \in \mathbb{D}^{+}$ with $0 \prec \epsilon \lesssim 2$, the conditions $\|\mathrm{u}\|_{\mathrm{B},\mathbb{D}} \lesssim 1$, $\|\mathrm{v}\|_{\mathrm{B},\mathbb{D}} \lesssim 1$, $\|\mathrm{u}-\mathrm{v}\|_{\mathrm{B},\mathbb{D}} \gtrsim \epsilon$ imply there exists a $\delta = \delta(\epsilon) \in \mathbb{D}^{+} \backslash \{0\}$ such that $\left\|\frac{\mathrm{u}+\mathrm{v}}{2}\right\|_{\mathrm{B},\mathbb{D}} \prec 1-\delta$ for all $\mathrm{u}, \mathrm{v} \in \mathrm{B}$.

Lemma 1.6. Consider $p \in \mathbb{R}$ with $p \in (1,2]$ and $q = \frac{p}{p-1}$. Assume that u and v are any two bicomplex numbers. Then we have

$$|u+v|_k^q + |u-v|_k^q \lesssim 2\left(|u|_k^p + |v|_k^p\right)^{q-1}.$$

Let us present a significant theorem provided in (Değirmen & Sağır, 2023). Also, for the definition and properties of the bicomplex $\mathbb{BC}$ −modules $l_p^k(\mathbb{BC})$, see (Değirmen & Sağır, 2023).

Theorem 1.7. Let $1 < p < \infty$. Then $l_p^k(\mathbb{BC})$ is $\mathbb{BC}$-strictly convex.

In the following, we briefly describe the properties and definitions of the weighted bicomplex sequence spaces $l_{p,\alpha}^k(\mathbb{BC})$ that have been extensively discussed in (Sağır & Güngör, 2023).

Definition 1.8. A bicomplex weighted sequence $\alpha = \{\alpha(n)\}_{n=0}^{\infty}$ is a sequence of positive hyperbolic numbers with $\alpha(0) = e_1 + e_2 = 1, \sum_{n=0}^{\infty} \alpha(n)$ is $\mathbb{D}$-divergent and $\alpha(n) \gtrsim 1$. We define the sets $l_{p,\alpha}^k(\mathbb{BC})$ for $0 < p < \infty$ and $l_{\infty,\alpha}^k(\mathbb{BC})$ utilizing the hyperbolic ($\mathbb{D}$-valued) norm $|.|_k$ in the following manner:

$$l_{p,\alpha}^k(\mathbb{BC}) := \left\{u = (u_n) \in s(\mathbb{BC}) : \sum_{n=0}^{\infty} |u_n|_k^p \alpha(n)\ \mathbb{D} - \text{converges}\right\}, 0 < p < \infty$$

and

$$l_{\infty,\alpha}^k(\mathbb{BC}) := \left\{u = (u_n) \in s(\mathbb{BC}) : \sup_{\mathbb{D}}\{|u_n|_k \alpha(n) : n \in \mathbb{N}\} \text{ is finite}\right\},$$

where $s(\mathbb{BC})$ is the $\mathbb{BC}$ −module of all bicomplex sequences. Let $u_n = u_{n1}e_1 + u_{n2}e_2, \alpha(n) = \alpha_1(n)e_1 + \alpha_2(n)e_2$. Then the complex series

$$\sum_{n=0}^{\infty} |u_{n1}|^p \alpha_1(n), \sum_{n=0}^{\infty} |u_{n2}|^p \alpha_2(n)$$

are convergent and so the sequences $\{u_{n1}\}_{n=0}^{\infty}$ and $\{u_{n2}\}_{n=0}^{\infty}$ belong to the weighted sequence spaces of complex numbers l_{p,α_1} and l_{p,α_2} respectively. Thus, $l_{p,\alpha}^k(\mathbb{BC})$ consists of all bicomplex sequences $u = (u_n) = (u_{n1}e_1 + u_{n2}e_2)$ such that $u_1 = (u_{n1}) \in l_{p,\alpha_1}$ and $u_2 = (u_{n2}) \in l_{p,\alpha_2}$ where $\alpha_1 = \alpha_1(n), \alpha_2 = \alpha_2(n)$ are complex weighted sequences. Therefore $l_{p,\alpha}^k(\mathbb{BC})$ can be written as

$$l_{p,\alpha}^k(\mathbb{BC}) = l_{p,\alpha_1}e_1 + l_{p,\alpha_2}e_2$$

Consider $u = (u_n) = (u_{n1}e_1 + u_{n2}e_2) \in l_{\infty,\alpha}^k(\mathbb{BC})$. Then the sequences $\{u_{n1}\}_{n=0}^{\infty}$ and $\{u_{n2}\}_{n=0}^{\infty}$ belong to the weighted bounded sequence spaces of complex numbers l_{∞,α_1} and l_{∞,α_2} respectively. As a result, $l_{\infty,\alpha}^k(\mathbb{BC})$ consists of all $\mathbb{D}$-bounded sequences $u = (u_n) = (u_{n1}e_1 + u_{n2}e_2)$ such that $u_1 = (u_{n1}) \in l_{\infty,\alpha_1}$ and $u_2 = (u_{n2}) \in l_{\infty,\alpha_2}$. Hence $l_{\infty,\alpha}^k(\mathbb{BC})$ can be written as

$$l_{\infty,\alpha}^k(\mathbb{BC}) = l_{\infty,\alpha_1}e_1 + l_{\infty,\alpha_2}e_2.$$

Proposition 1.9. The set $\mathrm{l}_{\infty,\alpha}^{\mathrm{k}}(\mathbb{BC})$ is a $\mathbb{BC}$-module under usual addition operation in sequences and bicomplex scalar multiplication.

Remark 1.10. Let $\{\alpha(\mathrm{n})\}_{\mathrm{n}=0}^{\infty}$ be a bicomplex weighted sequence. Consider $\mathrm{u} = (\mathrm{u_n}) \in \mathrm{l}_{\infty,\alpha}^{\mathrm{k}}(\mathbb{BC})$, where $\mathrm{u} = (\mathrm{u_n}) = (\mathrm{u_{n1}e_1} + \mathrm{u_{n2}e_2})$ and $\{\alpha(\mathrm{n})\}_{\mathrm{n}=0}^{\infty}$ are bicomplex weighted sequences. Clearly, the weighted spaces $\mathrm{l}_{\infty,\alpha_1}$ and $\mathrm{l}_{\infty,\alpha_2}$ are normed spaces with norms

$$\|u_1\|_{\infty,\alpha_1} = \sup\{|u_{n1}|\alpha_1(n): n \in \mathbb{N}\}$$

and

$$\|u_2\|_{\infty,\alpha_2} = \sup\{|u_{n2}|\alpha_2(n): n \in \mathbb{N}\}$$

respectively. Hence, we can endow the $\mathbb{BC}$-module $l^k_{\infty,\alpha}(\mathbb{BC})$ with hyperbolic ($\mathbb{D}$-valued) norm that is given by

$$\| u \|_{\mathbb{D},l_{\infty,\alpha}(\mathbb{BC})} = \|u_1\|_{\infty,\alpha_1} e_1 + \|u_2\|_{\infty,\alpha_2} e_2.$$

Let us take the function $\|.\|_{\mathbb{D},l^k_{\infty,\alpha}(\mathbb{BC})}$ that is defined as

$$\| u \|_{\mathbb{D},l^k_{\infty,\alpha}(\mathbb{BC})} = \sup_{\mathbb{D}}\{|u_n|_k \alpha(n): n \in \mathbb{N}\}.$$

Therefore, we have

$$\| u \|_{\mathbb{D},l^k_{\infty,\alpha}(\mathbb{BC})} = \| u \|_{l^k_{\infty,\alpha}(\mathbb{BC})} \quad (2)$$

for all $u \in l^k_{\infty,\alpha}(\mathbb{BC})$. Also, we can write

$$\| u \|_{l^k_{\infty,\alpha}(\mathbb{BC})} = \|\alpha_1 u_1\|_\infty e_1 + \|\alpha_2 u_2\|_\infty e_2$$

Remark 1.11. Let $\{\alpha(n)\}_{n=0}^{\infty}$ be a bicomplex weighted sequence. For $0 < p < \infty$, we have

$$l^k_{p,\alpha}(\mathbb{BC}) = l_{p,\alpha_1} e_1 + l_{p,\alpha_2} e_2$$

where $\alpha(n) = \alpha_1(n)e_1 + \alpha_2(n)e_2$, $u_n = u_{n1}e_1 + u_{n2}e_2$ such that l_{p,α_i} are complex weighted Lebesgue sequence spaces in sense

$$l_{p,\alpha_i} = \left\{u_i = (u_{ni}) \in s(\mathbb{C}): \sum_{n=0}^{\infty} |u_{ni}|^p \alpha_i(n) < \infty, i = 1,2.\right\}$$

Obviously, if $u = (u_n) \in l^k_{p,\alpha}(\mathbb{BC})$, then $\left(u_n \alpha(n)^{\frac{1}{p}}\right) \in l^k_p(\mathbb{BC})$.

Proposition 1.12. Let $\{\alpha(n)\}_{n=0}^{\infty}$ be a bicomplex weighted sequence. Then the set $l^k_{p,\alpha}(\mathbb{BC})$ for $0 < p < \infty$ is a $\mathbb{BC}$-submodule of $s(\mathbb{BC})$.

Remark 1.13. Let $\{\alpha(n)\}_{n=0}^{\infty}$ be a bicomplex weighted sequence. Consider $u \in l^k_{p,\alpha}(\mathbb{BC})$ with $u = (u_n) = (u_{n1}e_1 +$

$u_{n2}e_2)$, $\alpha = (\alpha(n)) = (\alpha_1(n)e_1 + \alpha_2(n)e_2)$, for $0 < p < \infty$. We may write

$$l_{p,\alpha}^k(\mathbb{BC}) = l_{p,\alpha_1}e_1 + l_{p,\alpha_2}e_2.$$

Obviously, the weighted Lebesgue sequence spaces l_{p,α_1} and l_{p,α_2} are normed spaces with

$$\|u_i\|_{l_{p,\alpha_i}} = \left(\sum_{n=0}^{\infty} |u_{ni}|^p \alpha_i(n)\right)^{1/p}, \qquad i = 1,2,$$

for $1 \le p < \infty$.

Therefore, we can endow the $\mathbb{BC}$-module $l_{p,\alpha}^k(\mathbb{BC})$ with hyperbolic ($\mathbb{D}$-valued) norm that is given by

$$\| u \|_{\mathbb{D},l_{p,\alpha}^k(\mathbb{BC})} = \|u_1\|_{l_{p,\alpha_1}} e_1 + \|u_2\|_{l_{p,\alpha_2}} e_2$$

where $u_1 = (u_{n1}), u_2 = (u_{n2}), \alpha_1 = \alpha_1(n), \alpha_2 = \alpha_2(n)$. Let us take the function $\| u \|_{l_{p,\alpha}^k(\mathbb{BC})}$ defined as

$$\| u \|_{l_{p,\alpha}^k(\mathbb{BC})} = \left(\sum_{n=0}^{\infty} |u_n|_k^p \alpha(n)\right)^{1/p}.$$

Hence, we obtain

$$\| u \|_{\mathbb{D},l_{p,\alpha}^k(\mathbb{BC})} = \| u \|_{l_{p,\alpha}^k(\mathbb{BC})} \tag{3}$$

for all $u \in l_{p,\alpha}^k(\mathbb{BC})$

Let $1 \le p < \infty$. Let us take $\alpha = \alpha_1 e_1 + \alpha_2 e_2$. Given that spaces $l_{p,\alpha}^k(\mathbb{BC})$ and $l_{\infty,\alpha}^k(\mathbb{BC})$ can be written as

$$l_{p,\alpha}^k(\mathbb{BC}) = l_{p,\alpha_1}e_1 + l_{p,\alpha_2}e_2$$

and

$$l_{\infty,\alpha}^k(\mathbb{BC}) = l_{\infty,\alpha_1}e_1 + l_{\infty,\alpha_2}e_2,$$

respectively, where l_{p,α_i} and l_{∞,α_i} are Banach spaces for $i = 1,2$, and equations (2) and (3) are considered, it is clear that normed spaces $l_{p,\alpha}^{k}(\mathbb{BC})$ and $l_{\infty,\alpha}^{k}(\mathbb{BC})$ are the bicomplex Banach modules with hyperbolic ($\mathbb{D}$-valued) norms from Theorem 3.5 in (Kumar & Singh, 2015) and Theorem 1.1 in (Kumar & et al., 2016).

Let $\{\alpha(n)\}_{n=0}^{\infty}$ be a bicomplex weighted sequence. This work demonstrates the $\mathbb{BC}$-convexity of $l_{p,\alpha}^{k}(\mathbb{BC})$ and $l_{\infty,\alpha}^{k}(\mathbb{BC})$ for $1 \leq p < \infty$. It is asserted that $l_{p,\alpha}^{k}(\mathbb{BC})$ is $\mathbb{BC}$-strictly convex for $1 < p < \infty$, and $\mathbb{BC}$-uniformly convex for $2 \leq p < \infty$. The $\mathbb{D}$-Hölder's and $\mathbb{D}$-Minkowski inequalities were stated for the case of $p \in (0,1)$. Furthermore, it is established that $l_{p}^{k}(\mathbb{BC})$ and $l_{p,\alpha}^{k}(\mathbb{BC})$ are $\mathbb{BC}$-uniformly convex for, $1 < p < 2$.

Main Results

Theorem 2.1. Let $\{\alpha(n)\}_{n=0}^{\infty}$ be a bicomplex weighted sequence and $1 \leq p < \infty$. Then $l_{p,\alpha}^{k}(\mathbb{BC})$ and $l_{\infty,\alpha}^{k}(\mathbb{BC})$ are $\mathbb{BC}$-convex.

Proof. In the beginning, let us demonstrate the $\mathbb{BC}$-convexity of $l_{p,\alpha}^{k}(\mathbb{BC})$ for $1 \leq p < \infty$. Let $\zeta = (\zeta_n), \eta = (\eta_n) \in l_{p,\alpha}^{k}(\mathbb{BC})$ and $\gamma \in \mathbb{D}^{+}$ where $0 \lesssim \gamma \lesssim 1$. Then $\sum_{n=0}^{\infty} |\zeta_n|_k^p \alpha(n)$ and $\sum_{n=0}^{\infty} |\eta_n|_k^p \alpha(n)$ coverges. Also, we have

$$\begin{aligned}
|\gamma\zeta_n + (1-\gamma)\eta_n|_k^p \alpha(n) &= \left|(\gamma\zeta_n + (1-\gamma)\eta_n)\alpha^{1/p}(n)\right|_k^p \\
&\lesssim \left(\left|\gamma\zeta_n\alpha^{1/p}(n)\right|_k + \left|(1-\gamma)\eta_n\alpha^{1/p}(n)\right|_k\right)^p \\
&\lesssim \left(2\sup_{\mathbb{D}}\left\{\left|\gamma\zeta_n\alpha^{1/p}(n)\right|_k, \left|(1-\gamma)\eta_n\alpha^{1/p}(n)\right|_k\right\}\right)^p \\
&= 2^p \sup_{\mathbb{D}}\left\{\left|\gamma\zeta_n\alpha^{1/p}(n)\right|_k^p, \left|(1-\gamma)\eta_n\alpha^{1/p}(n)\right|_k^p\right\} \\
&= 2^p \sup_{\mathbb{D}}\left\{\gamma^p\left|\zeta_n\alpha^{1/p}(n)\right|_k^p, (1-\gamma)^p\left|\eta_n\alpha^{1/p}(n)\right|_k^p\right\} \\
&\lesssim 2^p\left(\gamma^p\left|\zeta_n\alpha^{1/p}(n)\right|_k^p + (1-\gamma)^p\left|\eta_n\alpha^{1/p}(n)\right|_k^p\right)
\end{aligned}$$

for all $n = \mathbb{N} \cup \{0\}$. Hence we write

$$\sum_{n=0}^{\infty} |\gamma\zeta_n + (1-\gamma)\eta_n|_k^p \alpha(n)$$

$$\precsim 2^p \left(\gamma^p \sum_{n=0}^{\infty} \left|\zeta_n \alpha^{1/p}(n)\right|_k^p + (1-\gamma)^p \sum_{n=0}^{\infty} \left|\eta_n \alpha^{1/p}(n)\right|_k^p \right)$$

$$= 2^p \left(\gamma^p \sum_{n=0}^{\infty} |\zeta_n|_k^p \alpha(n) + (1-\gamma)^p \sum_{n=0}^{\infty} |\eta_n|_k^p \alpha(n) \right)$$

Therefore, based on the comparison test, it can be concluded that the series

$$\sum_{n=0}^{\infty} |\gamma\zeta_n + (1-\gamma)\eta_n|_k^p \alpha(n)$$

is convergent. Thus, $\gamma\zeta + (1-\gamma)\eta \in l_{p,\alpha}^k(\mathbb{BC})$ is achieved as intended. Hence, it can be concluded that $l_{p,\alpha}^k(\mathbb{BC})$ is $\mathbb{BC}$-convex.

Now, let us demonstrate the $\mathbb{BC}$-convexity of $l_{\infty,\alpha}^k(\mathbb{BC})$. Let $\zeta = (\zeta_n), \eta = (\eta_n) \in l_{\infty,\alpha}^k(\mathbb{BC})$ and $\gamma \in \mathbb{D}^+$ where $0 \precsim \gamma \precsim 1$. Then

$$\sup_{\mathbb{D}}\{|\zeta_n|_k \alpha(n) : n \in \mathbb{N}\}$$

and

$$\sup_{\mathbb{D}}\{|\eta_n|_k \alpha(n) : n \in \mathbb{N}\}$$

are finite. Therefore, we obtain

$$\sup_{\mathbb{D}}\{|\gamma\zeta_n+(1-\gamma)\eta_n|_k\alpha(n): n\in\mathbb{N}\}$$
$$\precsim \sup_{\mathbb{D}}\{\gamma|\zeta_n|_k\alpha(n)+(1-\gamma)|\eta_n|_k\alpha(n): n\in\mathbb{N}\}$$

$$=\gamma\sup_{\mathbb{D}}\{|\zeta_n|_k\alpha(n): n\in\mathbb{N}\}$$
$$+(1-\gamma)\sup_{\mathbb{D}}\{|\eta_n|_k\alpha(n): n\in\mathbb{N}\}.$$

Consequently, $\gamma\zeta+(1-\gamma)\eta\in l_{\infty,\alpha}^{k}(\mathbb{BC})$ is established. Hence, it can be concluded that $l_{\infty,\alpha}^{k}(\mathbb{BC})$ is $\mathbb{BC}$-convex.

Theorem 2.2. Let $\{\alpha(n)\}_{n=0}^{\infty}$ be a bicomplex weighted sequence. Then $l_{p,\alpha}^{k}(\mathbb{BC})$ is $\mathbb{BC}$-strictly convex for $1<p<\infty$.

Proof. Let $\zeta,\eta\in l_{p,\alpha}^{k}(\mathbb{BC})$ with $\zeta\neq\eta$, $\|\zeta\|_{\mathbb{D},l_{p,\alpha}^{k}(\mathbb{BC})}=\|\eta\|_{\mathbb{D},l_{p,\alpha}^{k}(\mathbb{BC})}=1$ and $\gamma\in\mathbb{D}^{+}$ with $0\precsim\gamma\precsim 1$. By Remark 1.11, we get $\zeta\alpha^{\frac{1}{p}}\in l_{p}^{k}(\mathbb{BC})$ and $\eta\alpha^{\frac{1}{p}}\in l_{p}^{k}(\mathbb{BC})$. Then we have

$$\left\|\zeta\alpha^{\frac{1}{p}}\right\|_{\mathbb{D},l_{p}^{k}(\mathbb{BC})}=1 \text{ and } \left\|\eta\alpha^{\frac{1}{p}}\right\|_{\mathbb{D},l_{p}^{k}(\mathbb{BC})}=1.$$

As stated by Theorem 1.7, it has been established that $l_{p}^{k}(\mathbb{BC})$ has $\mathbb{BC}$-strictly convexity for $1<p<\infty$. Therefore, we obtain

$$\|\gamma\zeta+(1-\gamma)\eta\|_{\mathbb{D},l_{p,\alpha}^{k}(\mathbb{BC})}=\left\|(\gamma\zeta+(1-\gamma)\eta)\alpha^{\frac{1}{p}}\right\|_{\mathbb{D},l_{p}^{k}(\mathbb{BC})}$$
$$=\left\|\gamma\zeta\alpha^{\frac{1}{p}}+(1-\gamma)\eta\alpha^{\frac{1}{p}}\right\|_{\mathbb{D},l_{p}^{k}(\mathbb{BC})}$$
$$\prec 1$$

for $1 < p < \infty$. Consequently, the property of $\mathbb{BC}$-strictly convexity is assured in $l^k_{p,\alpha}(\mathbb{BC})$ for $1 < p < \infty$.

Theorem 2.3. Let $\{\alpha(n)\}_{n=0}^{\infty}$ be a bicomplex weighted sequence. Then $l^k_{p,\alpha}(\mathbb{BC})$ is $\mathbb{BC}$-uniformly convex for $2 \leq p < \infty$.

Proof. Let $2 \leq p < \infty$. Let us take $\zeta = (\zeta_n), \eta = (\eta_n) \in l^k_{p,\alpha}(\mathbb{BC})$, $\epsilon \in \mathbb{D}^+$ with $0 \prec \epsilon \precsim 2$, $\|\zeta\|_{\mathbb{D},l^k_{p,\alpha}(\mathbb{BC})} \precsim 1$, $\|\eta\|_{\mathbb{D},l^k_{p,\alpha}(\mathbb{BC})} \precsim 1$, $\|\zeta - \eta\|_{\mathbb{D},l^k_{p,\alpha}(\mathbb{BC})} \succsim \epsilon$. Thus, we can write

$$\|\zeta + \eta\|^p_{\mathbb{D},l^k_{p,\alpha}(\mathbb{BC})} + \|\zeta - \eta\|^p_{\mathbb{D},l^k_{p,\alpha}(\mathbb{BC})} = \sum_{n=0}^{\infty} |\zeta_n + \eta_n|^p_k \alpha(n) + \sum_{n=0}^{\infty} |\zeta_n - \eta_n|^p_k \alpha(n)$$

$$= \sum_{n=0}^{\infty} \left(|\zeta_n + \eta_n|^p_k \alpha(n) + |\zeta_n - \eta_n|^p_k \alpha(n) \right)$$

$$= \sum_{n=0}^{\infty} \left(\left| (\zeta_n + \eta_n) \alpha^{\frac{1}{p}}(n) \right|^p_k + \left| (\zeta_n - \eta_n) \alpha^{\frac{1}{p}}(n) \right|^p_k \right).$$

By applying the inequality stated in Lemma 1.2, we have

$$\|\zeta + \eta\|^p_{\mathbb{D},l^k_{p,\alpha}(\mathbb{BC})} + \|\zeta - \eta\|^p_{\mathbb{D},l^k_{p,\alpha}(\mathbb{BC})} \precsim \sum_{n=0}^{\infty} 2^{p-1} \left| \zeta_n \alpha^{\frac{1}{p}}(n) \right|^p_k + \left| \eta_n \alpha^{\frac{1}{p}}(n) \right|^p_k$$

$$\|\zeta + \eta\|^p_{\mathbb{D},l^k_{p,\alpha}(\mathbb{BC})} + \|\zeta - \eta\|^p_{\mathbb{D},l^k_{p,\alpha}(\mathbb{BC})} \precsim 2^{p-1} \left(\sum_{n=0}^{\infty} |\zeta_n|^p_k \alpha(n) + \sum_{n=0}^{\infty} |\eta_n|^p_k \alpha(n) \right)$$

$$= 2^{p-1}\left(\|\zeta\|_{\mathbb{D},l_{p,\alpha}^k(\mathbb{BC})} + \|\eta\|_{\mathbb{D},l_{p,\alpha}^k(\mathbb{BC})}\right)$$

$$\precsim 2^{p-1}.2 = 2^p.$$

Therefore, we get

$$\|\zeta+\eta\|^p_{\mathbb{D},l_{p,\alpha}^k(\mathbb{BC})} \precsim 2^p - \|\zeta-\eta\|^p_{\mathbb{D},l_{p,\alpha}^k(\mathbb{BC})} \precsim 2^p - \epsilon^p,$$

it follows that

$$\left\|\frac{\zeta+\eta}{2}\right\|_{\mathbb{D},l_{p,\alpha}^k(\mathbb{BC})} = \left(\frac{1}{2^p}\|\zeta+\eta\|^p_{\mathbb{D},l_{p,\alpha}^k(\mathbb{BC})}\right)^{\frac{1}{p}} \precsim \left(1-\left(\frac{\epsilon}{2}\right)^p\right)^{\frac{1}{p}}.$$

If an assumption is made that $\delta(\epsilon) = 1 - \left(1-\left(\frac{\epsilon}{2}\right)^p\right)^{1/p}$, then it is possible to conclude that $l_{p,\alpha}^k(\mathbb{BC})$ is $\mathbb{BC}$-uniformly convex for $2 \le p < \infty$.

Lemma 2.4. ($\mathbb{D}$-Holder's inequality for $p \in (0,1)$) Let p and q be real numbers with $0 < \mathrm{p} < 1$ and $-\infty < \mathrm{p} < 0$ such that $\frac{1}{\mathrm{p}} + \frac{1}{\mathrm{q}} = 1$. Assume that $\zeta_{\mathrm{m}}, \eta_{\mathrm{m}} \in \mathbb{BC}$ for $\mathrm{m} \in \{1,2,\dots,\mathrm{n}\}$. Then we have

$$\sum_{m=1}^{n} |\zeta_m\eta_m|_k \succsim \left(\sum_{m=1}^{n} |\zeta_m|_k^p\right)^{\frac{1}{p}} \left(\sum_{m=1}^{n} |\eta_m|_k^q\right)^{\frac{1}{q}}.$$

Proof. Suppose that the conditions of the lemma hold. Let $\zeta_m = \zeta_{m1}e_1 + \zeta_{m2}e_2$ and $\eta_m = \eta_{m1}e_1 + \eta_{m2}e_2$ for $m \in \{1,2,\dots,n\}$. By using the properties of $|.|_k$, we can write

$$\sum_{m=1}^{n} |\zeta_m\eta_m|_k = \sum_{m=1}^{n} |\zeta_m|_k|\eta_m|_k$$

$$= \sum_{m=1}^{n} |\zeta_{m1}e_1 + \zeta_{m2}e_2|_k |\eta_{m1}e_1 + \eta_{m2}e_2|_k$$

$$= \sum_{m=1}^{n} (|\zeta_{m1}|e_1 + |\zeta_{m2}|e_2)(|\eta_{m1}|e_1 + |\eta_{m2}|e_2)$$

$$= \sum_{m=1}^{n} (|\zeta_{m1}||\eta_{m1}|e_1 + |\zeta_{m2}||\eta_{m2}|e_2)$$

$$= \left(\sum_{m=1}^{n} |\zeta_{m1}||\eta_{m1}|\right) e_1 + \left(\sum_{m=1}^{n} |\zeta_{m2}||\eta_{m2}|\right) e_2.$$

By applying the usual Hölder's inequality for $p \in (0,1)$ in Section 2.8 of (Hardy & et al., 1952), we get

$$\sum_{m=1}^{n} |\zeta_m \eta_m|_k \gtrsim \left(\left(\sum_{m=1}^{n} |\zeta_{m1}|^p\right)^{\frac{1}{p}} \left(\sum_{m=1}^{n} |\eta_{m1}|^q\right)^{\frac{1}{q}}\right) e_1$$

$$+ \left(\left(\sum_{m=1}^{n} |\zeta_{m2}|^p\right)^{\frac{1}{p}} \left(\sum_{m=1}^{n} |\eta_{m2}|^q\right)^{\frac{1}{q}}\right) e_2$$

$$= \left(\left(\sum_{m=1}^{n} |\zeta_{m1}|^p\right)^{\frac{1}{p}} e_1 + \left(\sum_{m=1}^{n} |\zeta_{m2}|^p\right)^{\frac{1}{p}} e_2\right)$$

$$\times\left(\left(\sum_{m=1}^{n}|\eta_{m1}|^{q}\right)^{\frac{1}{q}}e_1+\left(\sum_{m=1}^{n}|\eta_{m2}|^{q}\right)^{\frac{1}{q}}e_2\right).$$

By considering the property stated in Definition 1.1, we obtain

$$\sum_{m=1}^{n}|\zeta_m\eta_m|_k \gtrsim \left(\left(\sum_{m=1}^{n}|\zeta_{m1}|^{p}\right)e_1+\left(\sum_{m=1}^{n}|\zeta_{m2}|^{p}\right)e_2\right)^{\frac{1}{p}}$$

$$\times\left(\left(\sum_{m=1}^{n}|\eta_{m1}|^{q}\right)e_1+\left(\sum_{m=1}^{n}|\eta_{m2}|^{q}\right)e_2\right)^{\frac{1}{q}}$$

$$=\left(\sum_{m=1}^{n}|\zeta_{m1}|^{p}e_1+|\zeta_{m2}|^{p}e_2\right)^{\frac{1}{p}}\left(\sum_{m=1}^{n}|\eta_{m1}|^{q}e_1\right.$$

$$\left.+|\eta_{m2}|^{q}e_2\right)^{\frac{1}{q}}$$

$$=\left(\sum_{m=1}^{n}(|\zeta_{m1}|e_1+|\zeta_{m2}|e_2)^{p}\right)^{\frac{1}{p}}\left(\sum_{m=1}^{n}(|\eta_{m1}|e_1\right.$$

$$\left.+|\eta_{m2}|e_2)^{q}\right)^{\frac{1}{q}}$$

$$= \left(\sum_{m=1}^{n} |\zeta_{m1}e_1 + \zeta_{m2}e_2|_k^p\right)^{\frac{1}{p}} \left(\sum_{m=1}^{n} |\eta_{m1}e_1 + \eta_{m2}e_2|_k^q\right)^{\frac{1}{q}}$$

$$= \left(\sum_{m=1}^{n} |\zeta_m|_k^p\right)^{\frac{1}{p}} \left(\sum_{m=1}^{n} |\eta_m|_k^q\right)^{\frac{1}{q}}.$$

Lemma 2.5. ($\mathbb{D}$-Minkowski inequality for $p \in (0, 1)$) Let p be a real number with $0 < p < 1$. Assume that $\zeta_m, \eta_m \in \mathbb{BC}$ for $m \in \{1,2, \dots, n\}$. Then we have

$$\left(\sum_{m=1}^{n} |\zeta_m + \eta_m|_k^p\right)^{\frac{1}{p}} \gtrsim \left(\sum_{m=1}^{n} |\zeta_m|_k^p\right)^{\frac{1}{p}} + \left(\sum_{m=1}^{n} |\eta_m|_k^p\right)^{\frac{1}{p}}.$$

Proof. Suppose that the conditions of the lemma hold. Let $\zeta_m = \zeta_{m1}e_1 + \zeta_{m2}e_2$ and $\eta_m = \eta_{m1}e_1 + \eta_{m2}e_2$ for $m \in \{1,2, \dots, n\}$. By using the properties of $|.|_k$ and the property stated in Definition 1.1, we can write

$$\left(\sum_{m=1}^{n} |\zeta_m + \eta_m|_k^p\right)^{\frac{1}{p}}$$

$$= \left(\sum_{m=1}^{n} |(\zeta_{m1}e_1 + \zeta_{m2}e_2) + (\eta_{m1}e_1 + \eta_{m2}e_2)|_k^p\right)^{\frac{1}{p}}$$

$$= \left(\sum_{m=1}^{n} |(\zeta_{m1} + \eta_{m1})e_1 + (\zeta_{m2} + \eta_{m2})e_2|_k^p\right)^{\frac{1}{p}}$$

$$= \left(\sum_{m=1}^{n} |\zeta_{m1} + \eta_{m1}|^p e_1 + |\zeta_{m2} + \eta_{m2}|^p e_2 \right)^{\frac{1}{p}}$$

$$= \left(\left(\sum_{m=1}^{n} |\zeta_{m1} + \eta_{m1}|^p \right) e_1 + \left(\sum_{m=1}^{n} |\zeta_{m2} + \eta_{m2}|^p \right) e_2 \right)^{\frac{1}{p}}$$

$$= \left(\sum_{m=1}^{n} |\zeta_{m1} + \eta_{m1}|^p \right)^{\frac{1}{p}} e_1 + \left(\sum_{m=1}^{n} |\zeta_{m2} + \eta_{m2}|^p \right)^{\frac{1}{p}} e_2.$$

By applying the usual Minkowski inequality for $p \in (0,1)$ in Section 2.11 of (Hardy & et al., 1952), we get

$$\left(\sum_{m=1}^{n} |\zeta_m + \eta_m|_k^p \right)^{\frac{1}{p}} \gtrsim \left(\left(\sum_{m=1}^{n} |\zeta_{m1}|^p \right)^{\frac{1}{p}} + \left(\sum_{m=1}^{n} |\eta_{m1}|^p \right)^{\frac{1}{p}} \right) e_1$$

$$+ \left(\left(\sum_{m=1}^{n} |\zeta_{m2}|^p \right)^{\frac{1}{p}} + \left(\sum_{m=1}^{n} |\eta_{m2}|^p \right)^{\frac{1}{p}} \right) e_2$$

$$= \left(\left(\sum_{m=1}^{n} |\zeta_{m1}|^p \right)^{\frac{1}{p}} e_1 + \left(\sum_{m=1}^{n} |\zeta_{m2}|^p \right)^{\frac{1}{p}} e_2 \right)$$

$$+ \left(\left(\sum_{m=1}^{n} |\eta_{m1}|^p \right)^{\frac{1}{p}} e_1 \right.$$

$$\left. + \left(\sum_{m=1}^{n} |\eta_{m2}|^p \right)^{\frac{1}{p}} e_2 \right)$$

$$= \left(\left(\sum_{m=1}^{n} |\zeta_{m1}|^p \right) e_1 + \left(\sum_{m=1}^{n} |\zeta_{m2}|^p \right) e_2 \right)^{\frac{1}{p}}$$

$$+ \left(\left(\sum_{m=1}^{n} |\eta_{m1}|^p \right) e_1 + \left(\sum_{m=1}^{n} |\eta_{m2}|^p \right) e_2 \right)^{\frac{1}{p}}$$

$$\left(\sum_{m=1}^{n} |\zeta_m + \eta_m|_k^p \right)^{\frac{1}{p}}$$

$$\precsim \left(\sum_{m=1}^{n} |\zeta_{m1}|^p e_1 + |\zeta_{m2}|^p e_2 \right)^{\frac{1}{p}}$$

$$+ \left(\sum_{m=1}^{n} |\eta_{m1}|^p e_1 + |\eta_{m2}|^p e_2 \right)^{\frac{1}{p}}$$

$$= \left(\sum_{m=1}^{n} (|\zeta_{m1}| e_1 + |\zeta_{m2}| e_2)^p \right)^{\frac{1}{p}}$$

$$+ \left(\sum_{m=1}^{n} (|\eta_{m1}| e_1 + |\eta_{m2}| e_2)^p \right)^{\frac{1}{p}}$$

$$= \left(\sum_{m=1}^{n} |\zeta_{m1} e_1 + \zeta_{m2} e_2|_k^p \right)^{\frac{1}{p}}$$

$$+ \left(\sum_{m=1}^{n} |\eta_{m1} e_1 + \eta_{m2} e_2|_k^p \right)^{\frac{1}{p}}$$

$$= \left(\sum_{m=1}^{n} |\zeta_m|_k^p\right)^{\frac{1}{p}} + \left(\sum_{m=1}^{n} |\eta_m|_k^p\right)^{\frac{1}{p}}.$$

Proposition 2.6. Let p and q be real numbers with $1 < p < 2$ and $q = \frac{p}{p-1}$. Assume that $\zeta = (\zeta_n), \eta = (\eta_n) \in l_p^k(\mathbb{BC})$. Then we have

$$\|\zeta + \eta\|_{\mathbb{D},l_p^k(\mathbb{BC})}^q + \|\zeta - \eta\|_{\mathbb{D},l_p^k(\mathbb{BC})}^q \lesssim 2\left(\|\zeta\|_{\mathbb{D},l_p^k(\mathbb{BC})}^p + \|\eta\|_{\mathbb{D},l_p^k(\mathbb{BC})}^p\right)^{q-1}.$$

Proof. Suppose that the conditions of the theorem hold. Firstly, we can write

$$\|\zeta\|_{\mathbb{D},l_p^k(\mathbb{BC})}^q = \left(\sum_{n=1}^{\infty} |\zeta_n|_k^p\right)^{\frac{q}{p}} = \left(\sum_{n=1}^{\infty} |\zeta_n|_k^{q(p-1)}\right)^{\frac{1}{p-1}} = \left\||\zeta|_k^q\right\|_{\mathbb{D},l_{p-1,\alpha}^k(\mathbb{BC})}. \tag{4}$$

By applying the equality (4) to $\|\zeta + \eta\|_{\mathbb{D},l_p^k(\mathbb{BC})}^q$ and $\|\zeta - \eta\|_{\mathbb{D},l_p^k(\mathbb{BC})}^q$, respectively, and using $\mathbb{D}$-Minkowski inequality for $p - 1 \in (0,1)$, we get

$$\begin{aligned}\|\zeta + \eta\|_{\mathbb{D},l_p^k(\mathbb{BC})}^q + \|\zeta - \eta\|_{\mathbb{D},l_p^k(\mathbb{BC})}^q &= \left\||\zeta + \eta|_k^q\right\|_{\mathbb{D},l_{p-1}^k(\mathbb{BC})} + \left\||\zeta - \eta|_k^q\right\|_{\mathbb{D},l_{p-1}^k(\mathbb{BC})} \\ &\lesssim \left\||\zeta + \eta|_k^q + |\zeta - \eta|_k^q\right\|_{\mathbb{D},l_{p-1}^k(\mathbb{BC})}.\end{aligned}$$

By utilizing the expression provided in Lemma 1.6, we obtain

$$\|\zeta+\eta\|^q_{\mathbb{D},l^k_p(\mathbb{BC})}+\|\zeta-\eta\|^q_{\mathbb{D},l^k_p(\mathbb{BC})} \precsim 2\left\|\left(|\zeta|^p_k+|\eta|^p_k\right)^{q-1}\right\|_{\mathbb{D},l^k_{p-1}(\mathbb{BC})}$$

$$=2\left(\sum_{n=1}^{\infty}\left(|\zeta_n|^p_k+|\eta_n|^p_k\right)^{(q-1)(p-1)}\right)^{\frac{1}{p-1}}$$

$$=2\left(\sum_{n=1}^{\infty}\left(|\zeta_n|^p_k+|\eta_n|^p_k\right)\right)^{\frac{1}{p-1}}$$

$$=2\left(\|\zeta\|^p_{\mathbb{D},l^k_p(\mathbb{BC})}+\|\eta\|^p_{\mathbb{D},l^k_p(\mathbb{BC})}\right)^{q-1}.$$

Theorem 2.7. Let $1<\mathrm{p}<2$. Then $\mathrm{l}^{\mathrm{k}}_{\mathrm{p}}(\mathbb{BC})$ is $\mathbb{BC}$-uniformly convex.

Proof. Let $1<p<2$. Let us take $\zeta=(\zeta_n),\eta=(\eta_n)\in l^k_p(\mathbb{BC})$, $\epsilon\in\mathbb{D}^+$ with $0\prec\epsilon\precsim 2$, $\|\zeta\|_{\mathbb{D},l^k_p(\mathbb{BC})}\precsim 1$, $\|\eta\|_{\mathbb{D},l^k_p(\mathbb{BC})}\precsim 1$, $\|\zeta-\eta\|_{\mathbb{D},l^k_p(\mathbb{BC})}\succsim\epsilon$. By Proposition 2.6, we can write

$$\|\zeta+\eta\|^q_{\mathbb{D},l^k_p(\mathbb{BC})} \precsim 2\left(\|\zeta\|^p_{\mathbb{D},l^k_p(\mathbb{BC})}+\|\eta\|^p_{\mathbb{D},l^k_p(\mathbb{BC})}\right)^{q-1} - \|\zeta-\eta\|^q_{\mathbb{D},l^k_p(\mathbb{BC})}$$

$$\precsim 2^q-\epsilon^q=2^q\left(1-\left(\frac{\epsilon}{2}\right)^q\right).$$

Then, we have

$$\|\zeta+\eta\|^p_{\mathbb{D},l^k_p(\mathbb{BC})} \precsim 2^p\left(1-\left(\frac{\epsilon}{2}\right)^q\right)^{\frac{p}{q}}$$

Hence, it follows that

$$\left\|\frac{\zeta+\eta}{2}\right\|_{\mathbb{D},l^k_p(\mathbb{BC})} = \left(\frac{1}{2^p}\|\zeta+\eta\|^p_{\mathbb{D},l^k_p(\mathbb{BC})}\right)^{\frac{1}{p}} \precsim \left(1-\left(\frac{\epsilon}{2}\right)^q\right)^{\frac{1}{q}}.$$

If an assumption is made that $\delta(\epsilon) = 1-\left(1-\left(\frac{\epsilon}{2}\right)^q\right)^{1/q}$, then it is possible to conclude that $l^k_p(\mathbb{BC})$ is $\mathbb{BC}$-uniformly convex for $1 < p < 2$.

Proposition 2.8. Let $\{\alpha(n)\}_{n=0}^{\infty}$ be a bicomplex weighted sequence. Let p and q be real numbers with $1 < p \leq 2$ and $q = \frac{p}{p-1}$. Assume that $\zeta = (\zeta_n), \eta = (\eta_n) \in l^k_{p,\alpha}(\mathbb{BC})$. Then we have

$$\|\zeta+\eta\|^q_{\mathbb{D},l^k_{p,\alpha}(\mathbb{BC})} + \|\zeta-\eta\|^q_{\mathbb{D},l^k_{p,\alpha}(\mathbb{BC})} \precsim 2\left(\|\zeta\|^p_{\mathbb{D},l^k_{p,\alpha}(\mathbb{BC})} + \|\eta\|^p_{\mathbb{D},l^k_{p,\alpha}(\mathbb{BC})}\right)^{q-1}.$$

Proof. Let $v = (v_n), s = (s_n) \in l^k_p(\mathbb{BC})$. By $\mathbb{D}$-Minkowski inequality for $r \in (0,1)$, we can write

$$\left(\sum_{n=0}^{\infty}|v_n|^r_k\right)^{\frac{1}{r}} + \left(\sum_{n=0}^{\infty}|s_n|^r_k\right)^{\frac{1}{r}} \precsim \left(\sum_{n=0}^{\infty}|v_n+s_n|^r_k\right)^{\frac{1}{r}}. \quad (5)$$

Let $\{\alpha(n)\}_{n=0}^{\infty}$ be a bicomplex weighted sequence. Let p and q be real numbers with $1 < p \leq 2$ and $q = \frac{p}{p-1}$. Assume that $\zeta = (\zeta_n), \eta = (\eta_n) \in l^k_{p,\alpha}(\mathbb{BC})$. Thus $\zeta\alpha^{\frac{1}{p}}, \eta\alpha^{\frac{1}{p}} \in l^k_p(\mathbb{BC})$ by Remark 1.11. Let us replace r by $\frac{p}{q}$ and in (5) for

$$v_n = \left|(\zeta_n + \eta_n)\alpha^{\frac{1}{p}}(n)\right|_k^q$$

and

$$s_n = \left|(\zeta_n - \eta_n)\alpha^{\frac{1}{p}}(n)\right|_k^q.$$

Then we write

$$\left(\sum_{n=0}^{\infty} \left|(\zeta_n + \eta_n)\alpha^{\frac{1}{p}}(n)\right|_k^p\right)^{\frac{q}{p}} + \left(\sum_{n=0}^{\infty} \left|(\zeta_n - \eta_n)\alpha^{\frac{1}{p}}(n)\right|_k^p\right)^{\frac{q}{p}}$$

$$\lesssim \left(\sum_{n=0}^{\infty} \left(\left|(\zeta_n + \eta_n)\alpha^{\frac{1}{p}}(n)\right|_k^q + \left|(\zeta_n - \eta_n)\alpha^{\frac{1}{p}}(n)\right|_k^q\right)^{\frac{p}{q}}\right)^{\frac{q}{p}}$$

$$= \left(\sum_{n=0}^{\infty} \left(\left|\left(\zeta_n\alpha^{\frac{1}{p}}(n) + \eta_n\alpha^{\frac{1}{p}}(n)\right)\right|_k^q + \left|\left(\zeta_n\alpha^{\frac{1}{p}}(n) - \eta_n\alpha^{\frac{1}{p}}(n)\right)\right|_k^q\right)^{\frac{p}{q}}\right)^{\frac{q}{p}}.$$

By using Lemma 1.6, we get

$$\left(\sum_{n=0}^{\infty} \left|(\zeta_n + \eta_n)\alpha^{\frac{1}{p}}(n)\right|_k^p\right)^{\frac{q}{p}} + \left(\sum_{n=0}^{\infty} \left|(\zeta_n - \eta_n)\alpha^{\frac{1}{p}}(n)\right|_k^p\right)^{\frac{q}{p}}$$

$$\lesssim \left(\sum_{n=0}^{\infty} \left(2\left(\left|\zeta_n\alpha^{\frac{1}{p}}(n)\right|_k^p + \left|\eta_n\alpha^{\frac{1}{p}}(n)\right|_k^p\right)^{q-1}\right)^{\frac{p}{q}}\right)^{\frac{q}{p}}$$

$$= 2\left(\sum_{n=0}^{\infty}\left(\left|\zeta_n \alpha^{\frac{1}{p}}(n)\right|_k^p + \left|\eta_n \alpha^{\frac{1}{p}}(n)\right|_k^p\right)\right)^{\frac{q}{p}}$$

$$= 2\left(\sum_{n=0}^{\infty} |\zeta_n|_k^p \alpha(n) + \sum_{n=0}^{\infty} |\eta_n|_k^p \alpha(n)\right)^{\frac{q}{p}}$$

$$= 2\left(\|\zeta\|_{\mathbb{D},l_{p,\alpha}^k(\mathbb{BC})}^p + \|\eta\|_{\mathbb{D},l_{p,\alpha}^k(\mathbb{BC})}^p\right)^{q-1}$$

where $q-1=\frac{p}{q}$. Therefore, we obtain

$$\|\zeta+\eta\|_{\mathbb{D},l_{p,\alpha}^k(\mathbb{BC})}^q + \|\zeta-\eta\|_{\mathbb{D},l_{p,\alpha}^k(\mathbb{BC})}^q \precsim 2\left(\|\zeta\|_{\mathbb{D},l_{p,\alpha}^k(\mathbb{BC})}^p + \|\eta\|_{\mathbb{D},l_{p,\alpha}^k(\mathbb{BC})}^p\right)^{q-1}.$$

Theorem 2.9. Let $\{\alpha(n)\}_{n=0}^{\infty}$ be a bicomplex weighted sequence. Then $l_{p,\alpha}^k(\mathbb{BC})$ is $\mathbb{BC}$-uniformly convex for $1 < p < 2$.

Proof. Let $1 < p < 2$. Let us take $\zeta = (\zeta_n), \eta = (\eta_n) \in l_{p,\alpha}^k(\mathbb{BC})$, $\epsilon \in \mathbb{D}^+$ with $0 \prec \epsilon \precsim 2$, $\|\zeta\|_{\mathbb{D},l_{p,\alpha}^k(\mathbb{BC})} \precsim 1$, $\|\eta\|_{\mathbb{D},l_{p,\alpha}^k(\mathbb{BC})} \precsim 1$, $\|\zeta-\eta\|_{\mathbb{D},l_{p,\alpha}^k(\mathbb{BC})} \succsim \epsilon$. By Proposition 2.8, we can write

$$\|\zeta+\eta\|_{\mathbb{D},l_{p,\alpha}^k(\mathbb{BC})}^q \precsim 2\left(\|\zeta\|_{\mathbb{D},l_{p,\alpha}^k(\mathbb{BC})}^p + \|\eta\|_{\mathbb{D},l_{p,\alpha}^k(\mathbb{BC})}^p\right)^{q-1} - \|\zeta-\eta\|_{\mathbb{D},l_{p,\alpha}^k(\mathbb{BC})}^q$$

$$\precsim 2^q - \epsilon^q = 2^q\left(1-\left(\frac{\epsilon}{2}\right)^q\right).$$

Then, we have

$$\|\zeta+\eta\|^p_{\mathbb{D},l^k_{p,\alpha}(\mathbb{BC})} \lesssim 2^p\left(1-\left(\frac{\epsilon}{2}\right)^q\right)^{\frac{p}{q}}$$

Hence, it follows that

$$\left\|\frac{\zeta+\eta}{2}\right\|_{\mathbb{D},l^k_{p,\alpha}(\mathbb{BC})} = \left(\frac{1}{2^p}\|\zeta+\eta\|^p_{\mathbb{D},l^k_{p,\alpha}(\mathbb{BC})}\right)^{\frac{1}{p}} \lesssim \left(1-\left(\frac{\epsilon}{2}\right)^q\right)^{\frac{1}{q}}.$$

If an assumption is made that $\delta(\epsilon)=1-\left(1-\left(\frac{\epsilon}{2}\right)^q\right)^{1/q}$, then it is possible to conclude that $l^k_{p,\alpha}(\mathbb{BC})$ is $\mathbb{BC}$-uniformly convex for $1<p<2$.

REFERENCES

Alpay D., Luna-Elizarrarás, M. E., Shapiro, M. & Struppa, D. C. (2014). *Basics of functional analysis with bicomplex scalars, and bicomplex Schur analysis*. Springer Science & Business Media.

Değirmen N. & Sağır B. (2023). On bicomplex $\mathbb{BC}$-modules $l_p^k(\mathbb{BC})$ and some of their geometric properties. *Georgian Mathematical Journal*, 30 (1), 65-79.

Değirmen, N. & Sağır, B. (2021). $\mathbb{D}$ −Topological duals of bicomplex BC-modules $l_p^k(\mathbb{BC})$. *Journal of Mathematical Extension*, 16.

Gervais Lavoie R., Marchildon L. & Rochon D. (2010). Infinite-dimensional bicomplex Hilbert spaces. *Annals of Functional Analysis*, 1 (2), 75–91.

Gervais Lavoie R., Marchildon L. & Rochon D. (2011). Finite-dimensional bicomplex Hilbert spaces. *Advances in Applied Clifford Algebras*, 21 (3), 561–581.

Güngör, N. (2020). Some geometric properties of the non-Newtonian sequence spaces $l_p(N)$. *Mathematica Slovaca*, 70 (3), 689-696.

Hardy, G. H., Littlewood, J. E., & Pólya, G. (1952). *Inequalities* (Second edit). United Kingdom: Cambridge University Press.

Kumar R., Kumar R. & Rochon D. (2011). The fundamental theorems in the framework of bicomplex topological modules. *arXiv preprint* arXiv:1109.3424

Kumar R. & Singh K. (2015). Bicomplex linear operators on bicomplex Hilbert spaces and Littlewood's subordination theorem. *Advances in Applied Clifford Algebras*, 25 (3), 591–610.

Kumar R., Singh K., Saini H. & Kumar, S. (2016). Bicomplex weighted Hardy spaces and bicomplex C^*-algebras. *Advances in Applied Clifford Algebras*, 26 (1), 217–235.

Luna-Elizarraras M. E., Shapiro M, Struppa, D. C. & Vajiac, A. (2015). *Bicomplex holomorphic functions: The algebra, geometry and analysis of bicomplex numbers.* Basel: Frontiers in Mathematics, Birkhauser.

Oğur, O. (2019). Some geometric properties of weighted Lebesgue spaces $L^p_\omega(G)$. *Facta Universitatis, Series: Mathematics and Informatics*, 523-530.

Price G. B. (1991). *An introduction to multicomplex spaces and functions.* New York: Marcel Dekker Inc.

Rochon D. & Tremblay S. (2006). Bicomplex quantum mechanics: II. The Hilbert space. *Advances in Applied Clifford Algebras*, 2 (16), 135–157.

Sağır B. & Alşalvar, İ. (2019). On geometric properties of weighted Lebesgue sequence spaces. *Ikonion Journal of Mathematics*, 1 (1), 18-25.

Sağır, B. & Güngör, N. (2023). On Bicomplex Weighted $\mathbb{BC}$-Modules $l_p^k(\mathbb{BC})$ with Hyperbolıc-Valued Norm. B. Turgut (Ed.), *Reflections of Change in Engineering, Science, Maths and Education* (pp. 17-36). Klaipeda: SRA Academic Publishing.

Toksoy, E. & Sağır, B. (2023). On geometrical characteristics and inequalities of new bicomplex Lebesgue Spaces with hyperbolic-valued norm. *Georgian Mathematical Journal*. Doi:10.1515/gmj-2023-2093

CHAPTER V

Degree Based Topological Descriptors and Polynomials of Certain Dendrimers

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Elif ERYAŞAR[2]

Introduction

Nowadays, computer-aided design for productions in every field is a trend investigation area. Owing to this, it is possible to avoid from losts such as time and budget, arising from long and costly experimental processes. Using graph theory applications is one of the methods contributing to this aim.

Graph theory is related to the modeling of objects and relations of each object with each other by representing each object as a vertex and each relation of objects as an edge. The concept of topological index, which takes place in chemical graph theory, succeeds to

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obtain numeric values from molecular graphs of compounds, and these numerical datas have a big importance for the creation and efficiency of existing and also new designs. Every time it is not impossible to do computing stages manually via formulas. There are hundreds of topological indices and also various of them have complex formulas. To avoid from difficulties of calculations, there exist some methods applied via algebraic polynomials such as "M-polynomial", "CoM-polynomial" etc... (See from [1-3]) Each of them is a degree-based polynomial derivated by the edge and vertex partition technique from the graph and complement graph of a compound.

In [17] authors handled the optical transpose interconnection system swapped network via topological indices. In [18], polynomials of degree-based topological indices for the OTIS and swapped networks have been studied.

Dendrimers are multibranched composite structures that have been studied extensively [4-9] and are of many applications in drug delivery [10, 11], catalysis [12], and light harvesting [13]. Nanostar dendrimer is a part of the new group of macro-particles that appear to be photon funnels just like artificial antennas and are used in the formation of nanotubes, micro and macro-capsules, chemical sensors, colored glasses, and modified electrodes [14-16].

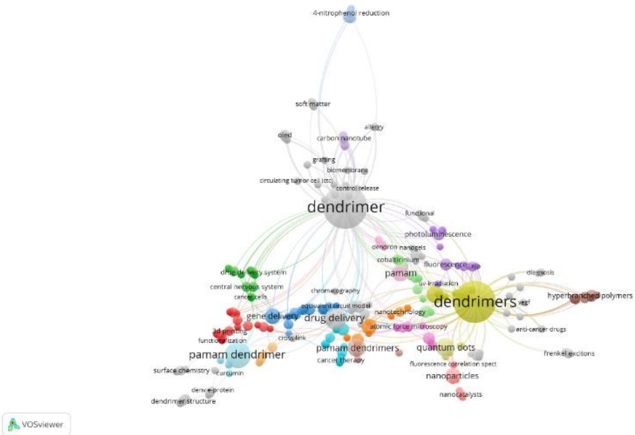


Figure 1. Bibliometric analysis of the keywords of dendrimers and applications in physics

The Bibliometric analysis of the keywords in publications of dendrimers in physics using the WOS data base (https://www.webofscience.com/wos/woscc/basic-search) is presented on in Figure 1.

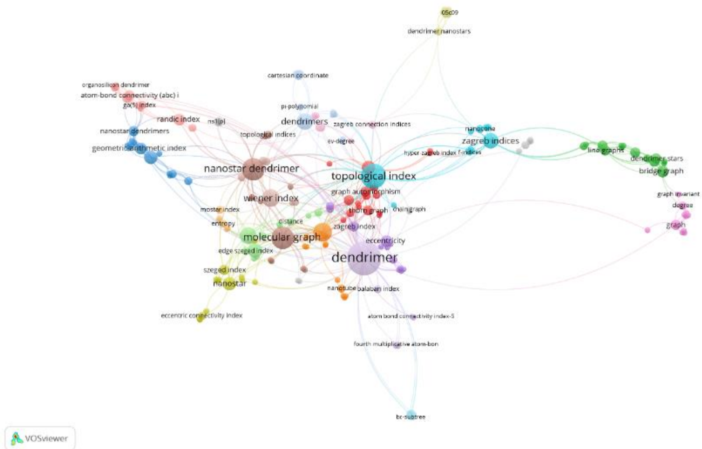


Figure 2. Bibliometric analysis of the keywords of topological indices and dendrimers

The Bibliometric analysis of the keywords in publications of dendrimers in different fields using the WOS data base (https://www.webofscience.com/wos/woscc/basic-search) is presented on in Figure 2.

Following the invention of dendrimers the possibility was recognized of using them for improving optical sensor performance [19]. Dendritic macromolecules are a new category of hyper-structured material and have recently been introduced into optical and optoelectronic applications. Here, functional chromophores can be replaced at branches, cores, and the end of the dendrimers to control their optical properties [20].

In this study, we calculate various Banhatti topological (co)indices, which are defined in the near past [21-24], for two dendrimers: Tetrathiafulvalene and Organosilicon dendrimers via two algebraic polynomials mentioned above. We give numerical and graphical comparisons to indicate the performance of indices and coindices. We hope these representations and numeric data may be helpful for testing the efficiency of optical applications of dendrimers in the future.

Preliminaries

The complement of a graph G, signed by $\bar{G}$, is a simple graph with the same vertex set $V(G)$ provided that any two vertices $v_1 v_2 \in E(\bar{G})$ if and only if $v_1 v_2 \notin E(G)$ [25]. Over time, researchers have begun to incorporate the nonadjacent pairing of vertices into consideration while calculating some topological indices of molecular graphs, resulting in degree-based topological indices known as coindices. In [3] Kirmani constructed CoM-polynomials as an alternative to M-polynomials using Berhe's Lemma [26] as follows:

Lemma 1. The following statement holds for a connected graph G of order n.

$$\bar{m}_{ij} = \left|\bar{E}_{ij}\right| = \begin{cases} \dfrac{n_i(n_i-1)}{2} - m_{ii} & , i = j \\ n_i n_j - m_{ij} & , \quad i < j \end{cases}$$

The following representations will be followed in the rest of the study for a graph G.

$n_i = |V_i|$ for $V_i = \{v \in V(G),\ d(v) = i\}$

$m_{ij} = \left|E_{ij}\right|$ for $E_{ij} = \{uv \in E(G),\ d(u) = i$ and $d(v) = j\}$

$\bar{m}_{ij} = \left|\bar{E}_{ij}\right|$ for $\bar{E}_{ij} = \{uv \in E(\bar{G}),\ d(u) = i$ and $n(v) = j\}$

where $d(u)$, $d(v)$ indicate the degrees of vertices that are adjacent to u and v, respectively.

Definition 1. For a simple connected graph G, M and CoM-polynomials are defined as,

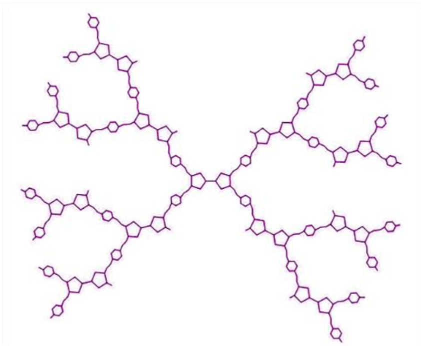
$$M(G; x, y) = \sum_{i \le j} m_{ij} x^i y^j$$

$$CoM(G; x, y) = \sum_{i \le j} \bar{m}_{ij} x^i y^j$$

where $\bar{m}_{ij}$ represents the number of edges $uv \notin E(G)$ such that $\{\left(d(u), d(v)\right) = \{i, j\}\}$.

Models and Methods

In this part of the study we will present some tables (Table 1 and Table 2) to construct our M-polynomials and CoM-polynomials by edge and vertex partition technique for our dendrimer molecules given in Figure 3.

a. 

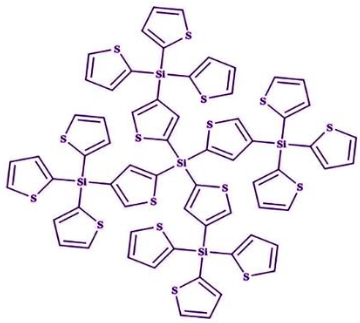
b. 

Figure 3. a. Tetrathiafulvalene dendrimer of generations G_n has grown 2 stages

b. The second member of the Organosilicon dendrimer structure

Tetrathiafulvalene Dendrimer

It has three type of vertices with the degrees 1, 2 and 3. Hence the vertex partition of the structure such that $n_1 = 6.2^n + 4, n_2 = 8.2^n + 8, n_3 = 30.2^n + 8$. *Using these, we obtain the following edge partition table (Table 1) by Lemma 1.*

Table 1. Edge partition for Tetrathiafulvalene dendrimer

$(\boldsymbol{d(u), d(v)});\ \boldsymbol{uv \in E(G)}$	**(1,3)**	**(2,1)**	**(2,2)**	**(2,3)**	**(3,3)**
$\lvert \boldsymbol{E_{ij}} \rvert = \boldsymbol{m_{ij}}$	$4.2^n - 4$	$8.2^n - 4$	$28.2^n - 16$	$92.2^n - 56$	$12.2^n - 9$
$\lvert \boldsymbol{\overline{E}_{ij}} \rvert = \boldsymbol{\overline{m}_{ij}}$	$180.2^{2n} + 164.2^n + 36$	$48.2^{2n} + 72.2^n + 36$	$32.2^{2n} + 32.2^n + 44$	$240.2^{2n} + 212.2^n + 120$	$450.2^{2n} + 213.2^n + 37$

Hence related polynomials for Tetrathiafulvalene dendrimer obtained as

$$M(x,y) = (4.2^n - 4)xy^3 + (8.2^n - 4)x^2y + (28.2^n - 16)x^2y^2 + (92.2^n - 56)x^2y^3 + (12.2^n - 9)x^3y^3$$

$$CoM(x,y) = (180.2^{2n} + 164.2^n + 36)xy^3 + (48.2^{2n} + 72.2^n + 36)x^2y + (32.2^{2n} + 32.2^n + 44)x^2y^2 + (240.2^{2n} + 212.2^n + 120)x^2y^3 + (450.2^{2n} + 213.2^n + 37)x^3y^3$$

Organosilicon Dendrimer

It has three type of vertices with the degrees 2, 3 and 4. Hence the vertex partition of the structure such that $n_2 = 2.3^n - 5, n_3 = \frac{10}{3}.3^n - 2, n_4 = \frac{16}{3}.3^n - 4$. *Using these, we obtain the following edge partition table (Table 2) by Lemma 1.*

Table 2. Edge partition for Organosilicon Dendrimer

$(d(u),d(v));\ uv \in E(G)$	$(2,2)$	$(2,3)$	$(3,4)$
$\lvert E_{ij}\rvert = m_{ij}$	$6.3^n - 6$	$4.3^n - 4$	$\frac{8}{3}.3^n - 4$
$\lvert \overline{E_{ij}}\rvert = \overline{m_{ij}}$	$2.3^{2n} - 17.3^n + 21$	$\frac{20}{3}.3^{2n} - \frac{74}{3}.3^n + 14$	$\frac{160}{9}.3^{2n} - \frac{80}{3}.3^n + 12$

Hence related polynomials for Tetrathiafulvalene dendrimer obtained as

$$M(x,y) = (6.3^n - 6)x^2y^2 + (4.3^n - 4)x^2y^3 + \left(\frac{8}{3}.3^n - 4\right)x^3y^4$$

$$CoM(x,y) = (2.3^{2n} - 17.3^n + 21)x^2y^2 + \left(\frac{20}{3}.3^{2n} - \frac{74}{3}.3^n + 14\right)x^2y^3 + \left(\frac{160}{9}.3^{2n} - \frac{80}{3}.3^n + 12\right)x^3y^4$$

Table 3. Topological indices

Topological Index	**Formula**	**Derivation From $M(G;\ x,y)$**
1. First K Banhatti Index ($B_1(G)$)	$\sum_{uv\in E(G)} (d_u + d_{uv})$	$[D_x + D_y + 2D_x Q_{-2} J](f(x,y))_{x=y=1}$
2. Second K Banhatti Index ($B_2(G)$)	$\sum_{uv\in E(G)} (d_u . d_{uv})$	$[D_x Q_{-2} J(D_x + D_y)](f(x,y))_{x=1}$
3. First K Hyper Banhatti Index ($HB_1(G)$)	$\sum_{uv\in E(G)} (d_u + d_{uv})^2$	$[D_x{}^2 + D_y{}^2 + 2D_x{}^2 Q_{-2} J + 2D_x Q_{-2} J(D_x + D_y)](f(x,y))_{x=y=1}$
4. Second K Hyper Banhatti Index ($HB_2(G)$)	$\sum_{uv\in E(G)} (d_u . d_{uv})^2$	$[D_x{}^2 Q_{-2} J(D_x{}^2 + D_y{}^2)](f(x,y))_{x=1}$
5. Modified First K Banhatti Index ($mB_1(G)$)	$\sum_{uv\in E(G)} \frac{1}{(d_u + d_{uv})}$	$[S_x Q_{-2} J(L_x + L_y)](f(x,y))_{x=1}$
6. Modified Second K Banhatti Index ($mB_2(G)$)	$\sum_{uv\in E(G)} \frac{1}{(d_u . d_{uv})}$	$[S_x Q_{-2} J(S_x + S_y)](f(x,y))_{x=1}$

Here motivated by Banhatti indices given in Table 3 we define various Banhatti coindices in Table 4. After this section we will calculate the related topological indices and coindices via derivations from our constructed M and CoM-polynomials.

Table 4. Topological Coindices

Topological Coindex	**Formula**	**Derivation From $CoM(G;\ x,y)$**
1. First K Banhatti Coindex ($\overline{B_1}(G)$)	$\sum_{uv\in E(\bar{G})} (d_u + d_{uv})$	$[D_x + D_y + 2D_x Q_{-2} J](f(x,y))_{x=y=1}$
2. Second K Banhatti Coindex ($\overline{B_2}(G)$)	$\sum_{uv\in E(\bar{G})} (d_u . d_{uv})$	$[D_x Q_{-2} J(D_x + D_y)](f(x,y))_{x=1}$
3. First K Hyper Banhatti Coindex ($\overline{HB_1}(G)$)	$\sum_{uv\in E(\bar{G})} (d_u + d_{uv})^2$	$[D_x{}^2 + D_y{}^2 + 2D_x{}^2 Q_{-2} J + 2D_x Q_{-2} J(D_x + D_y)](f(x,y))_{x=y=1}$
4. Second K Hyper Banhatti Coindex ($\overline{HB_2}(G)$)	$\sum_{uv\in E(\bar{G})} (d_u . d_{uv})^2$	$[D_x{}^2 Q_{-2} J(D_x{}^2 + D_y{}^2)](f(x,y))_{x=1}$
5. Modified First K Banhatti Coindex ($\overline{mB_1}(G)$)	$\sum_{uv\in E(\bar{G})} \frac{1}{(d_u + d_{uv})}$	$[S_x Q_{-2} J(L_x + L_y)](f(x,y))_{x=1}$

6. Modified Second K Banhatti Coindex ($\overline{mB_2}(G)$)	$\sum_{uv\in E(\bar{G})} \frac{1}{(d_u.d_{uv})}$	$[S_x Q_{-2} J(S_x + S_y)](f(x,y))_{x=1}$

$$D_x f(x,y) = x\frac{\partial(f(x,y))}{\partial x} \quad D_y f(x,y) = y\frac{\partial(f(x,y))}{\partial y}$$

$$L_x\big(f(x,y)\big) = f(x^2,y) \quad L_y\big(f(x,y)\big) = f(x,y^2)$$

$$S_x f(x,y) = \int_0^x \frac{f(t,y)}{t}dt \quad S_y f(x,y) = \int_0^y \frac{f(x,t)}{t}dt$$

$$Jf(x,y) = f(x,x) \quad Q_\alpha f(x,y) = x^\alpha f(x,y)$$

Computation Results

Computation of Topological Indices with the help of M-polynomials on Tetrathiafulvalene and Organosilicon Dendrimers

Theorem 1. The topological indices for Tetrathiafulvalene are given by,

$$B_1(G) = 1476.2^n - 922, \quad B_2(G) = 1948.2^n - 1236$$

$$HB_1(G) = 7900.2^n - 4990, \quad HB_2(G) = 15220.2^n - 9740$$

$$mB_1(G) = \frac{12592}{210}.2^n - \frac{1536}{42}, \quad mB_2(G) = \frac{1012}{18}.2^n - \frac{855}{18}.$$

Proof.

The M- polynomial of Tetrathiafulvalene is of the form

$$M(G;x,y) = (4.2^n - 4)xy^3 + (8.2^n - 4)x^2y + (28.2^n - 16)x^2y^2 + (92.2^n - 56)x^2y^3 + (12.2^n - 9)x^3y^3.$$

Then,

$$D_x = (4.2^n - 4)xy^3 + (16.2^n - 8)x^2y + (56.2^n - 32)x^2y^2 + (184.2^n - 112)x^2y^3 + (36.2^n - 27)x^3y^3$$

$$D_y = (12.2^n - 12)xy^3 + (8.2^n - 4)x^2y + (56.2^n - 32)x^2y^2 + (276.2^n - 168)x^2y^3 + (36.2^n - 27)x^3y^3$$

$$D_x + D_y = (16.2^n - 16)xy^3 + (24.2^n - 12)x^2y + (112.2^n - 64)x^2y^2 + (460.2^n - 280)x^2y^3 + (72.2^n - 54)x^3y^3$$

$$J = (8.2^n - 4)x^3 + (32.2^n - 20)x^4 + (92.2^n - 56)x^5 + (12.2^n - 9)x^6$$

$$2D_xQ_{-2}J = (16.2^n - 8)x^3 + (128.2^n - 80)x^4 + (552.2^n - 336)x^5 + (96.2^n - 72)x^6$$

$$D_xQ_{-2}J(D_x + D_y) = (24.2^n - 12)x^2 + (256.2^n - 160)x^2 + (1380.2^n - 840)x^3 + (288.2^n - 224)x^4$$

$$2D_xQ_{-2}J(D_x + D_y) = (48.2^n - 24)x^2 + (512.2^n - 320)x^2 + (2760.2^n - 1680)x^3 + (576.2^n - 448)x^4$$

$$2D_x^2Q_{-2}J = (16.2^n - 8)x^2 + (256.2^n - 160)x^2 + (1656.2^n - 1008)x^3 + (384.2^n - 288)x^4$$

$$D_x^2 = (4.2^n - 4)xy^3 + (32.2^n - 16)x^2y + (112.2^n - 64)x^2y^2 + (368.2^n - 224)x^2y^3 + (108.2^n - 81)x^3y^3$$

$$D_y^2 = (12.2^n - 12)xy^3 + (8.2^n - 4)x^2y + (112.2^n - 64)x^2y^2 + (828.2^n - 504)x^2y^3 + (108.2^n - 81)x^3y^3$$

$$D_x^2 + D_y^2 = (16.2^n - 16)xy^3 + (40.2^n - 20)x^2y + (224.2^n - 128)x^2y^2 + (1196.2^n - 728)x^2y^3 + (216.2^n - 162)x^3y^3$$

$$Q_{-2}J(D_x^2 + D_y^2)$$
$$= (40.2^n - 20)x + (240.2^n - 144)x^2 + (1196.2^n - 728)x^3 + (216.2^n - 162)x^4$$

$$D_x^2 Q_{-2}J(D_x^2 + D_y^2)$$
$$= (40.2^n - 20)x + (960.2^n - 576)x^2 + (10764.2^n - 6552)x^3 + (3456.2^n - 2592)x^4$$

$$L_x = (4.2^n - 4)x^2y^3 + (8.2^n - 4)x^4y + (28.2^n - 16)x^4y^2 + (92.2^n - 56)x^4y^3 + (12.2^n - 9)x^6y^3$$

$$L_y = (4.2^n - 4)xy^6 + (8.2^n - 4)x^2y^2 + (28.2^n - 16)x^2y^4 + (92.2^n - 56)x^2y^6 + (12.2^n - 9)x^3y^6$$

$$J(L_x + L_y) = (8.2^n - 4)x^4 + (12.2^n - 8)x^5 + (56.2^n - 32)x^6 + (96.2^n - 60)x^7 + (92.2^n - 56)x^8 + (24.2^n - 18)x^9$$

$$S_x Q_{-2}J(L_x + L_y)$$
$$= (4.2^n - 2)x^2 + \left(4.2^n - \frac{8}{3}\right)x^3 + (14.2^n - 8)x^4 + \left(\frac{96}{5}.2^n - 12\right)x^5 + \left(\frac{92}{6}.2^n - \frac{56}{6}\right)x^6 + \left(\frac{24}{7}.2^n - \frac{18}{7}\right)x^7$$

$$S_x = (4.2^n - 4)xy^3 + (4.2^n - 2)x^2y + (14.2^n - 8)x^2y^2 + (46.2^n - 28)x^2y^3 + (4.2^n - 3)x^3y^3$$

$$S_y = \left(\frac{4}{3}.2^n - \frac{4}{3}\right)xy^3 + (8.2^n - 4)x^2y + (14.2^n - 8)x^2y^2 + \left(\frac{92}{3}.2^n - \frac{56}{3}\right)x^2y^3 + (4.2^n - 3)x^3y^3$$

$$S_x + S_y = \left(\frac{16}{3}.2^n - \frac{16}{3}\right)xy^3 + (12.2^n - 6)x^2y + (28.2^n - 16)x^2y^2 + \left(\frac{230}{3}.2^n - \frac{140}{3}\right)x^2y^3 + (8.2^n - 6)x^3y^3$$

$$Q_{-2}J(S_x + S_y) = (12.2^n - 16)x + \left(\frac{100}{3}.2^n - \frac{100}{3}\right)x^2 + \left(\frac{230}{3}.2^n - \frac{140}{3}\right)x^3 + (8.2^n - 6)x^4$$

$$S_xQ_{-2}J(S_x + S_y) = (12.2^n - 16)x + \left(\frac{100}{6}.2^n - \frac{100}{6}\right)x^2 + \left(\frac{230}{9}.2^n - \frac{140}{9}\right)x^3 + \left(2.2^n - \frac{3}{2}\right)x^4.$$

Hence it is easy to calculate the given topological indices $x = 1 = y = 1$, as $B_1(G) = 1476.2^n - 922$, $B_2(G) = 1948.2^n - 1236$, $HB_1(G) = 7900.2^n - 4990$, $HB_2(G) = 15220.2^n - 9740$, $mB_1(G) = \frac{12592}{210}.2^n - \frac{1536}{42}$, $mB_2(G) = \frac{1012}{18}.2^n - \frac{855}{18}$.

Theorem 2. The topological indices for Organosilicon are given by,

$$B_1(G) = \frac{568}{3}.3^n - 232, \qquad B_2(G) = \frac{604}{3}.3^n - 248$$
$$HB_1(G) = \frac{1208}{3}.3^n - 496, \quad HB_2(G) = \frac{6980}{3}.3^n - 3160$$
$$mB_1(G) = \frac{554}{135}.3^n - \frac{487}{90}, \quad mB_2(G) = \frac{199}{45}.3^n - \frac{206}{45}.$$

Proof.

The M- polynomial of Organosilicon is of the form

$$M(G;x,y) = (6.3^n - 6)x^2y^2 + (4.3^n - 4)x^2y^3 + \left(\frac{8}{3}.3^n - 4\right)x^3y^4.$$

Then,

$$D_x = (12.3^n - 12)x^2y^2 + (8.3^n - 8)x^2y^3 + (8.3^n - 12)x^3y^4$$

$$D_y = (12.3^n - 12)x^2y^2 + (12.3^n - 12)x^2y^3 + \left(\frac{32}{3}.3^n - 16\right)x^3y^4$$

$$D_x + D_y = (24.3^n - 24)x^2y^2 + (20.3^n - 20)x^2y^3 + \left(\frac{56}{3}.3^n - 28\right)x^3y^4$$

$$J = (6.3^n - 6)x^4 + (4.3^n - 4)x^5 + \left(\frac{8}{3}.3^n - 4\right)x^7$$

$$2D_xQ_{-2}J = (24.3^n - 24)x^2 + (36.3^n - 36)x^3 + \left(\frac{200}{3}.3^n - 100\right)x^5$$

$$D_xQ_{-2}J\left(D_x + D_y\right) = (48.3^n - 48)x^2 + (60.3^n - 60)x^3 + \left(\frac{280}{3}.3^n - 140\right)x^5$$

$$2D_xQ_{-2}J(D_x + D_y) = (96.3^n - 96)x^2 + (120.3^n - 120)x^3 + \left(\frac{560}{3}.3^n - 280\right)x^5$$

$$2D_x^2Q_{-2}J = (48.3^n - 48)x^2 + (108.3^n - 108)x^3 + \left(\frac{1000}{3}.3^n - 500\right)x^5$$

$$D_x^2 = (24.3^n - 24)x^2y^2 + (16.3^n - 16)x^2y^3 + (24.3^n - 36)x^3y^4$$

$$D_y^2 = (24.3^n - 24)x^2y^2 + (36.3^n - 36)x^2y^3 + \left(\frac{128}{3}.3^n - 64\right)x^3y^4$$

$$D_x^2 + D_y^2 = (48.3^n - 48)x^2y^2 + (52.3^n - 52)x^2y^3 + \left(\frac{200}{3}.3^n - 100\right)x^3y^4$$

$$Q_{-2}J(D_x^2 + D_y^2) = (48.3^n - 48)x^2 + (52.3^n - 52)x^3 + \left(\frac{200}{3}.3^n - 100\right)x^5$$

$$D_x^2Q_{-2}J(D_x^2 + D_y^2) = (192.3^n - 192)x^2 + (468.3^n - 468)x^3 + \left(\frac{5000}{3}.3^n - 2500\right)x^5$$

$$L_x = (6.3^n - 6)x^4y^2 + (4.3^n - 4)x^4y^3 + \left(\frac{8}{3}.3^n - 4\right)x^6y^4$$

$$L_y = (6.3^n - 6)x^2y^4 + (4.3^n - 4)x^2y^6 + \left(\frac{8}{3}.3^n - 4\right)x^3y^8$$

$$J(L_x + L_y) = (12.3^n - 12)x^6 + (4.3^n - 4)x^7 + (4.3^n - 4)x^8 + \left(\frac{8}{3}.3^n - 4\right)x^{10} + \left(\frac{8}{3}.3^n - 4\right)x^{11}$$

$S_xQ_{-2}J(L_x + L_y)$

$$= (3.3^n - 3)x^4 + \left(\frac{4}{5}.3^n - \frac{4}{5}\right)x^5 + \left(\frac{2}{3}.3^n - \frac{2}{3}\right)x^6 + \left(\frac{1}{3}.3^n - \frac{1}{2}\right)x^8 + \left(\frac{8}{27}.3^n - \frac{4}{9}\right)x^9$$

$$S_x = (3.3^n - 3)x^2y^2 + (2.3^n - 2)x^2y^3 + \left(\frac{8}{9}.3^n - \frac{4}{3}\right)x^3y^4$$

$$S_y = (3.3^n - 3)x^2y^2 + \left(\frac{4}{3}.3^n - \frac{4}{3}\right)x^2y^3 + \left(\frac{2}{3}.3^n - 1\right)x^3y^4$$

$$S_x + S_y = (6.3^n - 6)x^2y^2 + \left(\frac{10}{3}.3^n - \frac{10}{3}\right)x^2y^3 + \left(\frac{14}{9}.3^n - \frac{7}{3}\right)x^3y^4$$

$$Q_{-2}J(S_x + S_y) = (6.3^n - 6)x^2 + \left(\frac{10}{3}.3^n - \frac{10}{3}\right)x^3 + \left(\frac{14}{9}.3^n - \frac{7}{3}\right)x^5$$

$S_xQ_{-2}J\left(S_x + S_y\right)$

$$= (3.3^n - 3)x^2 + \left(\frac{10}{9}.3^n - \frac{10}{9}\right)x^3 + \left(\frac{14}{45}.3^n - \frac{7}{45}\right)x^5.$$

Hence it is easy to calculate the given topological indices $x = 1 = y = 1$, as $B_1(G) = \frac{568}{3}.3^n - 232$, $B_2(G) = \frac{604}{3}.3^n - 248$, $HB_1(G) = \frac{1208}{3}.3^n - 496$, $HB_2(G) = \frac{6980}{3}.3^n - 3160$, $mB_1(G) = \frac{554}{135}.2^n - \frac{487}{90}$, $mB_2(G) = \frac{199}{45}.2^n - \frac{206}{45}$.

Computation of Topological Coindices with the help of CoM-Polynomials on Tetrathiafulvalene and Organosilicon Dendrimers

Theorem 3. The topological indices for Tetrathiafulvalene are given by,

$$\overline{B_1}(G) = 10876.2^{2n} + 7232.2^n + 2658, \qquad \overline{B_2}(G) = 16240.2^{2n} + 10$$

$$\overline{HB_1}(G) = 67560.2^{2n} + 42009.2^n + 14807, \qquad \overline{HB_2}(G) = 144544.2^{2n} + 9$$

$$\overline{mB_1}(G) = \frac{5835}{14}.2^{2n} + \frac{102387}{420}.2^n + \frac{18163}{140}, \qquad \overline{mB_2}(G) = \frac{1049}{3}.2^{2n} + \frac{5}{}$$

Proof.

The CoM- polynomial of Tetrathiafulvalene is of the form

$$\begin{aligned} CoM(G;x,y) = {} & (180.2^{2n} + 164.2^n + 36)xy^3 \\ & + (48.2^{2n} + 72.2^n + 36)x^2y \\ & + (32.2^{2n} + 32.2^n + 44)x^2y^2 \\ & + (240.2^{2n} + 212.2^n + 120)x^2y^3 \\ & + (450.2^{2n} + 213.2^n + 37)x^3y^3. \end{aligned}$$

Then,

$$\begin{aligned} D_x = {} & (180.2^{2n} + 164.2^n + 36)xy^3 \\ & + (96.2^{2n} + 144.2^n + 72)x^2y \\ & + (64.2^{2n} + 64.2^n + 88)x^2y^2 \\ & + (480.2^{2n} + 414.2^n + 240)x^2y^3 \\ & + (1350.2^{2n} + 639.2^n + 111)x^3y^3 \end{aligned}$$

$$\begin{aligned} D_y = {} & (540.2^{2n} + 492.2^n + 108)xy^3 \\ & + (48.2^{2n} + 72.2^n + 36)x^2y \\ & + (64.2^{2n} + 64.2^n + 88)x^2y^2 \\ & + (720.2^{2n} + 636.2^n + 360)x^2y^3 \\ & + (1350.2^{2n} + 639.2^n + 111)x^3y^3 \end{aligned}$$

$$D_x + D_y = (720.2^{2n} + 656.2^n + 144)xy^3 + (144.2^{2n} + 216.2^n + 108)x^2y + (128.2^{2n} + 128.2^n + 176)x^2y^2 + (1200.2^{2n} + 1050.2^n + 600)x^2y^3 + (2700.2^{2n} + 1278.2^n + 222)x^3y^3$$

$$J = (48.2^{2n} + 72.2^n + 36)x^3 + (212.2^{2n} + 196.2^n + 80)x^4 + (240.2^{2n} + 212.2^n + 120)x^5 + (450.2^{2n} + 213.2^n + 37)x^6$$

$$2D_xQ_{-2}J = (96.2^{2n} + 144.2^n + 72)x + (848.2^{2n} + 784.2^n + 320)x^2 + (1440.2^{2n} + 1272.2^n + 720)x^3 + (3600.2^{2n} + 1704.2^n + 296)x^4$$

$$D_xQ_{-2}J(D_x + D_y) = (144.2^{2n} + 216.2^n + 108)x + (1696.2^{2n} + 1568.2^n + 640)x^2 + (3600.2^{2n} + 3150.2^n + 1800)x^3 + (10800.2^{2n} + 5112.2^n + 888)x^4$$

$$2D_xQ_{-2}J(D_x + D_y) = (288.2^{2n} + 432.2^n + 216)x + (3392.2^{2n} + 3136.2^n + 1280)x^2 + (7200.2^{2n} + 6300.2^n + 3600)x^3 + (21600.2^{2n} + 10224.2^n + 1776)x^4$$

$$2D_x^2Q_{-2}J = (192.2^{2n} + 288.2^n + 144)x + (1696.2^{2n} + 1568.2^n + 720)x^2 + (4320.2^{2n} + 3816.2^n + 2160)x^3 + (14400.2^{2n} + 6816.2^n + 1184)x^4$$

$$D_x^2 = (180.2^{2n} + 164.2^n + 36)xy^3$$
$$+ (192.2^{2n} + 288.2^n + 144)x^2y$$
$$+ (128.2^{2n} + 128.2^n + 176)x^2y^2$$
$$+ (960.2^{2n} + 828.2^n + 480)x^2y^3$$
$$+ (2700.2^{2n} + 1278.2^n + 222)x^3y^3$$

$$D_y^2 = (1620.2^{2n} + 1476.2^n + 324)xy^3$$
$$+ (48.2^{2n} + 72.2^n + 36)x^2y$$
$$+ (128.2^{2n} + 128.2^n + 176)x^2y^2$$
$$+ (2160.2^{2n} + 3150.2^n + 1800)x^2y^3$$
$$+ (4050.2^{2n} + 1917.2^n + 333)x^3y^3$$

$$D_x^2 + D_y^2 = (1800.2^{2n} + 1640.2^n + 360)xy^3$$
$$+ (240.2^{2n} + 360.2^n + 180)x^2y$$
$$+ (256.2^{2n} + 256.2^n + 352)x^2y^2$$
$$+ (3120.2^{2n} + 3978.2^n + 2280)x^2y^3$$
$$+ (6750.2^{2n} + 3195.2^n + 555)x^3y^3$$

$$Q_{-2}J(D_x^2 + D_y^2)$$
$$= (240.2^{2n} + 360.2^n + 180)x$$
$$+ (2056.2^{2n} + 1896.2^n + 712)x^2$$
$$+ (3120.2^{2n} + 3978.2^n + 2280)x^3$$
$$+ (6750.2^{2n} + 3195.2^n + 555)x^4$$

$$D_x^2Q_{-2}J(D_x^2 + D_y^2)$$
$$= (240.2^{2n} + 360.2^n + 180)x$$
$$+ (8224.2^{2n} + 7584.2^n + 2848)x^2$$
$$+ (28080.2^{2n} + 35802.2^n + 20520)x^3$$
$$+ (108000.2^{2n} + 51120.2^n + 8880)x^4$$

$$L_x = (180.2^{2n} + 164.2^n + 36)x^2y^3 + (48.2^{2n} + 72.2^n + 36)x^4y + (32.2^{2n} + 32.2^n + 44)x^4y^2 + (240.2^{2n} + 212.2^n + 120)x^4y^3 + (450.2^{2n} + 213.2^n + 37)x^6y^3$$

$$L_y = (180.2^{2n} + 164.2^n + 36)xy^6 + (48.2^{2n} + 72.2^n + 36)x^2y^2 + (32.2^{2n} + 32.2^n + 44)x^2y^4 + (240.2^{2n} + 212.2^n + 120)x^2y^6 + (450.2^{2n} + 213.2^n + 37)x^3y^6$$

$$J(L_x + L_y) = (48.2^{2n} + 72.2^n + 36)x^4 + (228.2^{2n} + 236.2^n + 72)x^5 + (514.2^{2n} + 277.2^n + 125)x^6 + (420.2^{2n} + 376.2^n + 156)x^7 + (240.2^{2n} + 212.2^n + 120)x^8 + (450.2^{2n} + 213.2^n + 37)x^9$$

$$S_xQ_{-2}J(L_x + L_y) = (24.2^{2n} + 36.2^n + 18)x^2 + \left(76.2^{2n} + \frac{236}{3}.2^n + 24\right)x^3 + \left(\frac{257}{2}2^{2n} + \frac{277}{4}.2^n + \frac{125}{4}\right)x^4 + \left(84.2^{2n} + \frac{376}{5}.2^n + \frac{156}{5}\right)x^5 + \left(40.2^{2n} + \frac{212}{6}.2^n + 20\right)x^6 + \left(\frac{450}{7}.2^{2n} + \frac{213}{7}.2^n + \frac{37}{7}\right)x^7$$

$$S_x = (180.2^{2n} + 164.2^n + 36)xy^3$$
$$+ (24.2^{2n} + 36.2^n + 18)x^2y$$
$$+ (16.2^{2n} + 16.2^n + 22)x^2y^2$$
$$+ (120.2^{2n} + 106.2^n + 60)x^2y^3$$
$$+ \left(150.2^{2n} + 71.2^n + \frac{37}{3}\right)x^3y^3$$

$$S_y = \left(60.2^{2n} + \frac{164}{3}.2^n + 12\right)xy^3$$
$$+ (48.2^{2n} + 72.2^n + 36)x^2y$$
$$+ (16.2^{2n} + 16.2^n + 22)x^2y^2$$
$$+ \left(80.2^{2n} + \frac{212}{3}.2^n + 40\right)x^2y^3$$
$$+ \left(150.2^{2n} + 71.2^n + \frac{37}{3}\right)x^3y^3$$

$$S_x + S_y = \left(240.2^{2n} + \frac{656}{3}.2^n + 48\right)xy^3$$
$$+ (72.2^{2n} + 108.2^n + 54)x^2y$$
$$+ (32.2^{2n} + 32.2^n + 44)x^2y^2$$
$$+ \left(200.2^{2n} + \frac{530}{3}.2^n + 100\right)x^2y^3$$
$$+ \left(300.2^{2n} + 142.2^n + \frac{74}{3}\right)x^3y^3$$

$$Q_{-2}J(S_x + S_y) = (72.2^{2n} + 108.2^n + 54)x^2$$
$$+ \left(272.2^{2n} + \frac{752}{3}.2^n + 92\right)x^2$$
$$+ \left(200.2^{2n} + \frac{530}{3}.2^n + 100\right)x^3$$
$$+ \left(300.2^{2n} + 142.2^n + \frac{74}{3}\right)x^4$$

$$S_xQ_{-2}J(S_x+S_y)$$
$$= (72.2^{2n} + 108.2^n + 54)x + \left(136.2^{2n} + \frac{376}{3}.2^n + 46\right)x^2 + \left(\frac{200}{3}.2^{2n} + \frac{530}{9}.2^n + \frac{100}{3}\right)x^3 + \left(75.2^{2n} + \frac{71}{2}.2^n + \frac{37}{6}\right)x^4x^4.$$

Hence it is easy to calculate the given topological indices $x = 1 = y = 1$, as $\overline{B_1}(G) = 10876.2^{2n} + 7232.2^n + 2658$, $\overline{B_2(G)} = 16240.2^{2n} + 10046.2^n + 3436$, $\overline{HB_1}(G) = 67560.2^{2n} + 42009.2^n + 14807$ $\overline{HB_2}(G) = 144544.2^{2n} + 94866.2^n + 32428$, $\overline{mB_1}(G) = \frac{5835}{14}.2^{2n} + \frac{102387}{420}.2^n + \frac{18163}{140}$, $\overline{mB}_2(G) = \frac{1049}{3}.2^{2n} + \frac{5899}{18}.2^n + \frac{279}{2}$.

Theorem 4. The topological coindices for Organosilicon are given by,

$$\overline{B_1}(G) = 3524.3^{2n-2} + 2582.3^{n-1} + 526, \qquad \overline{B_2}(G) = 6644.3^{2n-2}$$
$$\overline{HB_1}(G) = 27436.3^{2n-2} - 17542.3^{n-1} + 3182, \qquad \overline{HB_2}(G) = 107596.3^{2n-2}$$
$$\overline{mB_1}(G) = 619.3^{2n-4} + \frac{6437}{10}.3^{n-3} + \frac{277}{15}, \qquad \overline{mB_2}(G) = 187.3^{2n-2} -$$

Proof.

The CoM- polynomial of Organosilicon is of the form

$$CoM(G;x,y) = (2.3^{2n} - 17.3^n + 21)x^2y^2 + \left(\frac{20}{3}.3^{2n} - \frac{74}{3}.3^n + 14\right)x^2y^3 + \left(\frac{160}{9}.3^{2n} - \frac{80}{3}.3^n + 12\right)x^3y^4.$$

Then,

$$D_x = (4.3^{2n} - 34.3^n + 42)x^2y^2 + \left(\frac{40}{3}.3^{2n} - \frac{148}{3}.3^n + 28\right)x^2y^3 + \left(\frac{480}{9}.3^{2n} - 80.3^n + 36\right)x^3y^4$$

$$D_y = (4.3^{2n} - 34.3^n + 42)x^2y^2 + (20.3^{2n} - 74.3^n + 42)x^2y^3 + \left(\frac{640}{9}.3^{2n} - \frac{320}{3}.3^n + 48\right)x^3y^4$$

$$D_x + D_y = (8.3^{2n} - 68.3^n + 84)x^2y^2 + \left(\frac{100}{3}.3^{2n} - \frac{370}{3}.3^n + 70\right)x^2y^3 + \left(\frac{1120}{9}.3^{2n} - \frac{5600}{3}.3^n + 84\right)x^3y^4$$

$$J = (2.3^{2n} - 17.3^n + 21)x^4 + \left(\frac{20}{3}.3^{2n} - \frac{74}{3}.3^n + 14\right)x^5 + \left(\frac{160}{9}.3^{2n} - \frac{80}{3}.3^n + 12\right)x^7$$

$$2D_xQ_{-2}J = (8.3^{2n} - 68.3^n + 84)x^2 + \left(40.3^{2n} - \frac{148}{3}.3^n + 84\right)x^3 + \left(\frac{1600}{9}.3^{2n} - \frac{800}{3}.3^n + 120\right)x^5$$

$$D_xQ_{-2}J(D_x + D_y) = (16.3^{2n} - 136.3^n + 168)x^2 + (100.3^{2n} - 370.3^n + 210)x^3 + \left(\frac{5600}{9}.3^{2n} - \frac{2800}{3}.3^n + 420\right)x^5$$

$$2D_xQ_{-2}J\left(D_x + D_y\right)$$
$$= (32.3^{2n} - 272.3^n + 336)x^2$$
$$+ (200.3^{2n} - 740.3^n + 420)x^3$$
$$+ \left(\frac{11200}{9}.3^{2n} - \frac{5600}{3}.3^n + 840\right)x^5$$

$$2D_x^2Q_{-2}J = (16.3^{2n} - 68.3^n + 84)x^2$$
$$+ (120.3^{2n} - 444.3^n + 252)x^3$$
$$+ \left(\frac{8000}{9}.3^{2n} - \frac{4000}{3}.3^n + 600\right)x^5$$

$$D_x^2 = (8.3^{2n} - 68.3^n + 84)x^2y^2$$
$$+ \left(\frac{80}{3}.3^{2n} - \frac{296}{3}.3^n + 56\right)x^2y^3$$
$$+ \left(\frac{1440}{9}.3^{2n} - 240.3^n + 108\right)x^3y^4$$

$$D_y^2 = (8.3^{2n} - 68.3^n + 84)x^2y^2$$
$$+ (60.3^{2n} - 222.3^n + 126)x^2y^3$$
$$+ \left(\frac{2560}{9}.3^{2n} - \frac{1280}{3}.3^n + 192\right)x^3y^4$$

$$D_x^2 + D_y^2 = (16.3^{2n} - 136.3^n + 168)x^2y^2$$
$$+ \left(\frac{260}{3}.3^{2n} - \frac{962}{3}.3^n + 182\right)x^2y^3$$
$$+ \left(\frac{4000}{9}.3^{2n} - \frac{2000}{3}.3^n + 300\right)x^3y^4$$

$$Q_{-2}J(D_x^2 + D_y^2)$$
$$= (16.3^{2n} - 136.3^n + 168)x^2$$
$$+ \left(\frac{260}{3}.3^{2n} - \frac{962}{3}.3^n + 182\right)x^3$$
$$+ \left(\frac{4000}{9}.3^{2n} - \frac{2000}{3}.3^n + 300\right)x^5$$

$$D_x^2 Q_{-2} J(D_x^2 + D_y^2)$$
$$= (64.3^{2n} - 544.3^n + 672)x^2$$
$$+ (780.3^{2n} - 2886.3^n + 546)x^3$$
$$+ \left(\frac{100000}{9}.3^{2n} - \frac{500000}{3}.3^n + 7500\right)x^5$$

$$L_x = (2.3^{2n} - 17.3^n + 21)x^4y^2$$
$$+ \left(\frac{20}{3}.3^{2n} - \frac{74}{3}.3^n + 14\right)x^4y^3$$
$$+ \left(\frac{160}{9}.3^{2n} - \frac{80}{3}.3^n + 12\right)x^6y^4$$

$$L_y = (2.3^{2n} - 17.3^n + 21)x^2y^4$$
$$+ \left(\frac{20}{3}.3^{2n} - \frac{74}{3}.3^n + 14\right)x^2y^6$$
$$+ \left(\frac{160}{9}.3^{2n} - \frac{80}{3}.3^n + 12\right)x^3y^8$$

$$J(L_x + L_y) = (4.3^{2n} - 34.3^n + 42)x^6$$
$$+ \left(\frac{20}{3}.3^{2n} - \frac{74}{3}.3^n + 14\right)x^7$$
$$+ \left(\frac{20}{3}.3^{2n} - \frac{74}{3}.3^n + 14\right)x^8$$
$$+ \left(\frac{160}{9}.3^{2n} - \frac{80}{3}.3^n + 12\right)x^{10}$$
$$+ \left(\frac{160}{9}.3^{2n} - \frac{80}{3}.3^n + 12\right)x^{11}$$

$$S_xQ_{-2}J(L_x + L_y)$$

$$= \left(3.3^{2n} - \frac{34}{3}.3^n + \frac{42}{4}\right)x^4 + \left(\frac{4}{3}.3^{2n} - \frac{74}{15}.3^n + \frac{14}{5}\right)x^7 + \left(\frac{10}{3}.3^{2n} - \frac{74}{18}.3^n + \frac{7}{3}\right)x^8 + \left(\frac{20}{9}.3^{2n} - \frac{10}{3}.3^n + \frac{3}{2}\right)x^{10} + \left(\frac{160}{81}.3^{2n} - \frac{80}{27}.3^n + \frac{4}{3}\right)x^{11}$$

$$S_x = \left(3.3^{2n} - \frac{17}{2}.3^n + \frac{21}{2}\right)x^2y^2 + \left(\frac{10}{3}.3^{2n} - \frac{37}{3}.3^n + 7\right)x^2y^3 + \left(\frac{160}{27}.3^{2n} - \frac{80}{9}.3^n + 4\right)x^3y^4$$

$$S_y = \left(3.3^{2n} - \frac{17}{2}.3^n + \frac{21}{2}\right)x^2y^2 + \left(\frac{20}{9}.3^{2n} - \frac{74}{9}.3^n + \frac{14}{3}\right)x^2y^3 + \left(\frac{40}{9}.3^{2n} - \frac{20}{3}.3^n + 3\right)x^3y^4$$

$$S_x + S_y = (6.3^{2n} - 17.3^n + 21)x^2y^2 + \left(\frac{50}{9}.3^{2n} - \frac{185}{9}.3^n + \frac{35}{3}\right)x^2y^3 + \left(\frac{280}{27}.3^{2n} - \frac{140}{9}.3^n + 7\right)x^3y^4$$

$Q_{-2}J(S_x+S_y)$

$$= (6.3^{2n} - 17.3^{n} + 21)x^2 + \left(\frac{50}{9}.3^{2n} - \frac{185}{9}.3^{n} + \frac{35}{3}\right)x^3 + \left(\frac{280}{27}.3^{2n} - \frac{140}{9}.3^{n} + 7\right)x^5$$

$S_xQ_{-2}J(S_x+S_y)$

$$= \left(3.3^{2n} - \frac{17}{2}.3^{n} + 21\right)x^2 + \left(\frac{50}{27}.3^{2n} - \frac{185}{27}.3^{n} + \frac{35}{9}\right)x^3 + \left(\frac{56}{27}.3^{2n} - \frac{28}{9}.3^{n} + \frac{7}{5}\right)x^5$$

Hence it is easy to calculate the given topological indices $x = 1 = y = 1$, as $\overline{B_1}(G) = 3524.3^{2n-2} + 2582.3^{n-1} + 526$, $\overline{B_2}(G) = 6644.3^{2n-2} - 4318.3^{n-1} + 798$, $\overline{HB_1}(G) = 27436.3^{2n-2} - 17542.3^{n-1} + 3182$, $\overline{HB_2}(G) = 107596.3^{2n-2} - 60290.3^{n-1} + 8718$, $\overline{mB_1}(G) = 619.3^{2n-4} - \frac{6437}{10}.3^{n-3} + \frac{277}{15}$, $\overline{mB_2}(G) = 187.3^{2n-3} - \frac{997}{2}.3^{n-3} + \frac{1421}{90}$.

Numerical and Graphical Comparison

In this section we aim to show the relation between related Banhatti indices and coindices percebtibly, via numerical and graphical comparison figures in Figure 4 and Figure 5. It is seen that in each graphic, new defined Banhatti topological coindices increase/decreases rapidly than existing Banhatti topological indices. For Tetrathiafulvalene dendrimer, each graphic has increasing curvilinear model for both related indices and coindices. But for Organosilicon dendrimer, while only (modified) first K Banhatti indices and coindices have been incresing at the same time, the other ones are of decreased coindex lines while indices are of increased lines. These results are of guiding significance to the engineering application.

Figure 4. Numerical and Graphical Comparison of Organosilicon

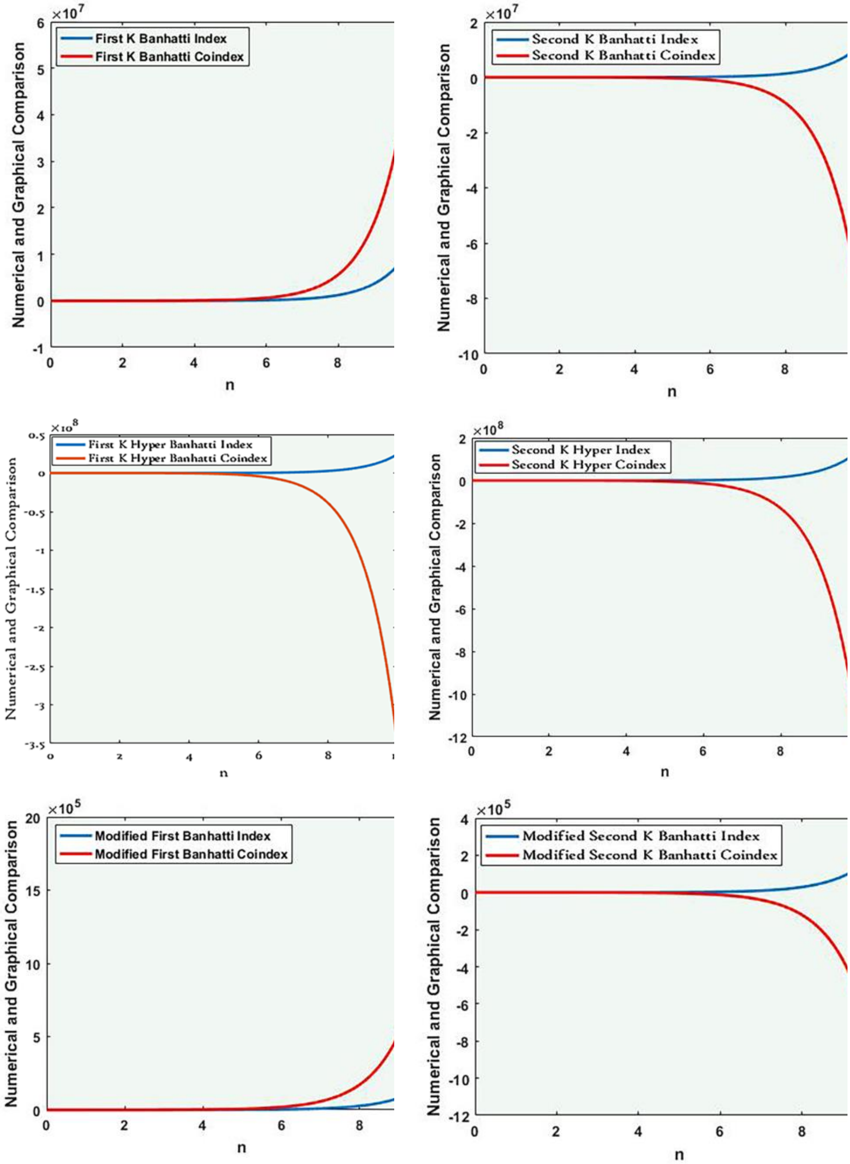
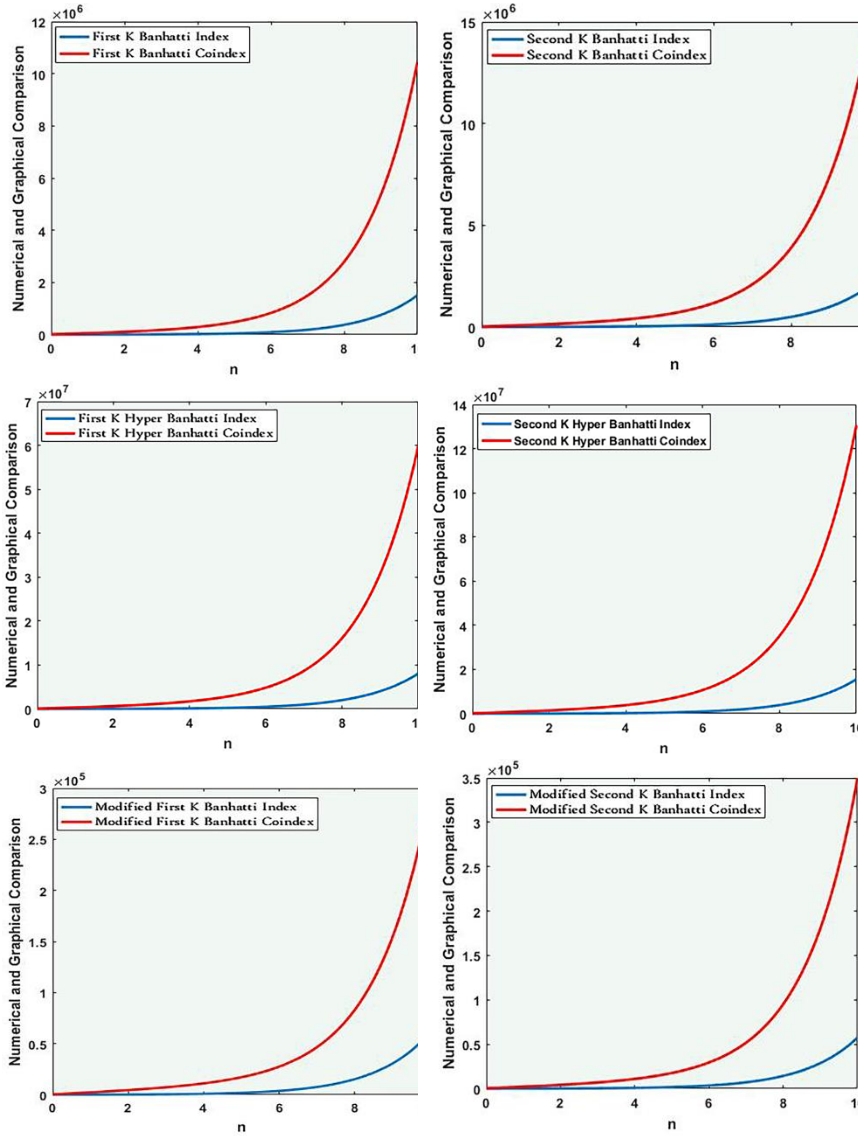


Figure 5. Numerical and Graphical Comparison of Tetrathiafulvalene Dendrimer



6. Summarization and Conclusion

Topological indices are widely used calculation tools to predict physicochemical and bioactivity properties of chemical compunds. They have many applications in medical and material sciences via QSPR/QSAR analysis to deisgn high performanced, low cost products. Nowadays researchers have been interested in studies on optical applications via topological approach. Also, dendrimers are investigated as an effective conductor because of its more branched structure. Motivated by these, in this work we handle six types of Banhatti indices and we define topological coindex versions via complement graph theory. Afterward, we produce the M and CoM-polynomials of two specific dentritic compounds from molecular graphs of each compund via vertex and each partition technique. By algebraic polynomial approach, we compute all indices and coindices. At the end we give comparisonal figures showing that the performance between topological indices and coindices. We hope these representations and numerical datas may be helpful for testing the efficiency of optical applications of dendrimers in the future.

REFERENCES

Deutsch, E. & Klavžar, S. (2015) M-polynomial and degree-based topological indices. *Iranian Journal of Mathematical Chemistry, 6(*2), 93-102. Doi: 10.48550/arXiv.1407.1592

Chaudary, F. Ehsan, M. Afzal, F. Farahani, M. R. Cancan, M. & Çiftçi, İ. (2021) Degree based topological indices of tadpole graph via M-Polynomial, *Eurasian Chem. Commun*. *3* (1) 146-153. Doi: 10.22034/ecc.2021.269116.1125

Kirmani, S.A.K. Ali, & P. Ahmad, J. (2022) Topological coindices and quantitative structure-property analysis of antiviral drugs investigated in the treatment of COVID-19. *Journal of Chemisty*, Doi: 10.1155/2022/3036655

Xavier, D. A. Akhila, S. Alsinai, A. Julietraja, K. Ahmed, H. Raja, A.A. & Varghese, E.S. (2022), Distance-based structure characterization of pamam-related dendrimers nanoparticle. *Journal of Nanomaterials*, 1-16. Doi: 10.1155/2022/2911196

Chu, Y. M. Siddiqui, M. K. & Nasir, M. (2022) On topological coindices of polycyclic Tetrathiafulvalene and polycyclic Organosilicon dendrimers. *Polycyclic Aromatic Compounds*, *42* (5), 2179-2197. Doi: 10.1080/10406638.2020.1830130

Imran, M. Hayat, S. & Shafiq, M. K. (2015) Valency based topological indices of Organosilicon Dendrimers and cactus chains. *Optoelectronics and Advanced Materials-Rapid Communications*, *9* (5-6), 821-830.

Nakayama, J. & Lin, J. S. (1997) An organosilicon dendrimer composed of 16 thiophene rings. *Tetrahedron Letters*, *38* (34), 6043-6046. Doi: 10.1016/S0040-4039(97)01356-7

Shah, A. & Bokhary, S.A.U.H. (2019) On chromatic polynomial of certain families of dendrimer graphs. *Open Journal of Mathematical Sciences*, *3*(1), 404-416. Doi: 10.30538/oms2019.0083

Bokhary, S.A.U.H. Imran, M. & Manzoor, S. (2016) On molecular topological properties of dendrimers. *Canadian Journal of Chemistry*, *94*(2), 120-125. Doi: 10.1139/cjc-2015-0466

Janaszewska, A. Lazniewska, J. Trzepiński, P. Marcinkowska, M. & Klajnert-Maculewicz, B. (2019) Cytotoxicity of dendrimers, *Biomolecules*, *9*(8), 330. Doi: 10.3390/biom9080330

Sherje, A. P. Jadhav, M. Dravyakar, B. R. & Kadam, D. (2018) Dendrimers: A versatile nanocarrier for drug delivery and targeting. *International Journal of Pharmaceutics*, *548*(1), 707-720. Doi: 10.1016/j.ijpharm.2018.07.030

Yamamoto, K. Imaoka, T. Tanabe, M. & Kambe, T. (2019) New horizon of nanoparticle and cluster catalysis with dendrimers, *Chemical Reviews*, *120*(2), 1397-1437. Doi: 10.1021/acs.chemrev.9b00188

El Kadib, A. Katir, N. Bousmina, M. & Majoral, J. P. (2012) Dendrimer–silica hybrid mesoporous materials. *New Journal of Chemistry*, *36*(2), 241-255. Doi: 10.1039/C1NJ20443B

Ashrafi, A. R. & Mirzargar, M. (2008) PI, Szeged and edge Szeged indices of an infinite family of nanostar dendrimers, *Indian Journal of Chemistry*, *47*, 538-541.

Diudea, M. V. & Katona, G. (1999) Advances in Dendritic Macromolecules.

Munir, M. Nazeer, W. Rafique, S. & Kang, S. M. (2016) M-polynomial and related topological indices of nanostar dendrimers. *Symmetry*, *8*(9), 97. Doi: 10.3390/sym8090097

Zahra, N. Ibrahim, M. & Siddiqui, M. K. (2020) On topological indices for swapped networks modeled by optical transpose interconnection system. *IEEE Access*, *8*, 200091-200099. Doi: 10.1109/ACCESS.2020.3034439

Ahmad, A. Hasni, R. Elahi, K. & Asim, M. A. (2020) Polynomials of degree-based indices for swapped networks modeled by optical transpose interconnection system. *IEEE Access*, *8*, 214293-214299. Doi: 10.1109/ACCESS.2020.3039298

Soršak, E. Valh, J. V. Urek, Š. K. & Lobnik, A. (2015) Application of PAMAM dendrimers in optical sensing. *Analyst*, *140*(4), 976-989. Doi: 10.1039/c4an00825a

Yokoyama, S. Otomo, A. Nakahama, T. Okuno, Y. & Mashiko, S. (2003) Dendrimers for optoelectronic applications. *Dendrimers V: Functional and Hyperbranched Building Blocks, Photophysical Properties, Applications in Materials and Life Sciences*, 205-226. Doi: 10.1007/b11012

Kulli, V. R. (2016) On K Banhatti indices of graphs. *Journal of Computer and Mathematical Sciences*, *7*(4), 213-218.

Kulli, V. R. & On, K. (2016) On K hyper-Banhatti indices and coindices of graphs. *International Research Journal of Pure Algebra*, *6*(5), 300-304.

Kulli, V. R. (2017) New K Banhatti topological indices. *International Journal of Fuzzy Mathematical Archive*, *12*(1), 29-37. Doi: 10.22457/ijfma.v12n1a4

Afzal, D. Ali, S. Afzal, F. Cancan, M. Ediz, S. & Farahani, M. R. (2021) A study of newly defined degree-based topological indices via M-polynomial of Jahangir graph, *Journal of Discrete Mathematical Sciences and Cryptography*, *24*(2), 427-438. Doi: 10.1080/09720529.2021.1882159

Trinajstic, N. (2018). Chemical graph theory. Routledge.

Berhe, M. & Wang, C. (2019) Computation of certain topological coindices of graphene sheet and () nanotubes and nanotorus. *Applied Mathematics and Nonlinear Sciences*, *4*(2), 455-468. Doi: 10.2478/AMNS.2019.2.00043

CHAPTER VI

On the Higher Order Leonardo Quaternions

Kübra GÜL[1]

Introduction

Number sequences have an important place and their applications in various scientific fields in the literature (Abrate, Barbero, Cerruti, & Murru, 2014; Amannah & Nanwin, 2014; Falcon, & Plaza, 2007; Koshy, 2001; Shannon, Deveci & Erdağ, 2019). The most famous of these sequences are the Fibonacci sequences $\{F_n\}_{n=0}^{\infty}$ and Lucas sequences $\{L_n\}_{n=0}^{\infty}$ defined by, $n \geq 2$,

$$F_n = F_{n-1} + F_{n-2}$$

and

$$L_n = L_{n-1} + L_{n-2}$$

, where $F_0 = 0$, $F_1 = 1$ and $L_0 = 2$, $L_1 = 1$, respectively.

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In (Catarino & Borges, 2019), Leonardo sequence $\{\mathcal{L}_n\}_{n=0}^{\infty}$ is defined by

$$\mathcal{L}_n = \mathcal{L}_{n-1} + \mathcal{L}_{n-2} + 1 \text{ n} \geq 2, \mathcal{L}_0 = \mathcal{L}_1 = 1.$$

Also, Leonardo sequence is given by the equation

$$\mathcal{L}_{n+1} = 2\mathcal{L}_n - \mathcal{L}_{n-2}.$$

The characteristic equation of the equation (4) is

$$\lambda^3 - 2\lambda^2 + 1 = 0.$$

The Binet's formula of the Leonardo sequence is

$$\mathcal{L}_n = \frac{2\alpha^{n+1} - 2\beta^{n+1} - \alpha + \beta}{\alpha - \beta} \tag{1}$$

where α and β are roots of characteristic equation.

There are the relationships between Leonardo numbers, Fibonacci numbers and Lucas numbers as follows:

$$\mathcal{L}_n = 2F_{n+1} - 1,$$
$$\mathcal{L}_{n-1} + \mathcal{L}_{n+1} = 2L_{n+1} - 2,$$
$$\mathcal{L}_n + 2F_n = \mathcal{L}_{n+1},$$
$$\mathcal{L}_n + F_n + L_n = 2\mathcal{L}_n + 1,$$
$$\mathcal{L}_{n+1}^2 + \mathcal{L}_n^2 = 2(\mathcal{L}_{2n+2} - \mathcal{L}_{n+2} + 1).$$

Several identities for Leonardo numbers were obtained by authors. Furthermore, they provided Leonardo numbers with a matrix representation. Some recent resarch on Leonardo numbers can be seen in (Alp & Koçer, 2021; Catarino & Borges, 2019).

In recent years, several authors investigated on higher order sequences associated with the well known number sequences. The study of higher order sequences began with the earlier work of Randi´c et al. (Randić, Morales & Araujo, 1996) where the authors investigated higher order Fibonacci numbers and its various algebraic properties. Cook et al. (Cook & Bacon, 2013) defined Jacobsthal higher order numbers. In (Prasad, Kumari, Mohanta, & Mahato, 2023), the authors introduced Mersenne higher order numbers and gave various algebraic properties. In the literature, there are several studies on higher order sequences associated with the diferent sequences and their generalizations. Gül (Gül, 2023) introduced higher order Leonardo numbers. Kizilateş and Kone (Kizilates & Kone, 2021a) expanded the quaternion algebraic research to higher order Fibonacci numbers. Then, the authors invastigated hyper complex numbers and quaternions with the higher order sequences (Gül, 2022; Kizilates & Kone, 2021b; Ozimamoğlu, 2023; Özkan & Uysal, 2023).

Quaternions were investigated by Hamilton (Hamilton, 1866) as an extension to the complex numbers. A quaternion is defined by

$$q = a_0 + a_1 \mathbb{i} + a_2 \mathbb{j} + a_3 \mathbb{k} \tag{2}$$

where a_0, a_1, a_2, a_3 are real numbers and $\mathbb{i}$, $\mathbb{j}$, $\mathbb{k}$ are quaternionic units which satisfy the following rules:

$$\mathbb{i}^2 = \mathbb{j}^2 = \mathbb{k}^2 = \mathbb{i}\mathbb{j}\mathbb{k} = -1 \text{ and } \mathbb{i}\mathbb{j} = \mathbb{k} = -\mathbb{j}\mathbb{i}, \mathbb{j}\mathbb{k} = \mathbb{i} = -\mathbb{k}\mathbb{j}, \mathbb{k}\mathbb{i} = \mathbb{j} = -\mathbb{i}\mathbb{k}. \tag{3}$$

Several authors studied on different quaternions and their generalizations, some of which can be found in (Asci & Aydinyuz, 2021; Gül, 2019; Gül, 2020; Halici, 2012 Horadam, 1963). Paulı Leonardo quaternion was defined by Leonardo numbers coefficients (Isbilir & Tosun, 2023). Nurkan et al. (Nurkan & Güven, 2023) introduced ordered Leonardo quadruple numbers by dual quaternions and Leonardo numbers. Then, Yılmaz et al. (Yılmaz & Saçlı, 2023) studied the dual quaternions with the k-generalized Leonardo sequence.

In this study, we introduce a new type of quaternions, called the higher order Leonardo quaternions. We consider the coefficients of these quaternions as the higher order Leonardo numbers. We give Binet formula, generating function, several identities for this sequence.

Main Results

For positive integer k, the higher order Leonardo numbers $\left\{\mathcal{L}_n^{(k)}\right\}$ are defined by

$$\mathcal{L}_n^{(k)} = \frac{\mathcal{L}_{kn}}{\mathcal{L}_k}, \qquad n = 0,1,2 \dots . \tag{4}$$

In (Gül, 2023), the Binet's formula of the higher order Leonardo numbers $\mathcal{L}_n^{(k)}$ can be written as

$$\mathcal{L}_n^{(k)} = \frac{2\alpha^{kn+1} - 2\beta^{kn+1} - \alpha + \beta}{2\alpha^{k+1} - 2\beta^{k+1} - \alpha + \beta}. \quad (5)$$

In (Mangueira, Alves & Catarino, 2022), Leonardo quaternion, denoted by $\mathcal{L}Q_n$, are defined by

$$\mathcal{L}Q_n = \mathcal{L}_n + \mathcal{L}_{n+1}\mathbb{i} + \mathcal{L}_{n+2}\mathbb{j} + \mathcal{L}_{n+3}\mathbb{k}.$$

The Binet's formula of the Leonardo quaternions

$$\mathcal{L}Q_n = \frac{2\alpha^{n+1}\widehat{\alpha} - 2\beta^{n+1}\widehat{\beta} + w(-\alpha + \beta)}{\alpha - \beta} \quad (6)$$

where $\hat{\alpha} = 1 + \alpha^k\mathbb{i} + \alpha^{2k}\mathbb{j} + \alpha^{3k}\mathbb{k}$, $\hat{\beta} = 1 + \beta^k\mathbb{i} + \beta^{2k}\mathbb{j} + \beta^{3k}\mathbb{k}$ and $w = 1 + \mathbb{i} + \mathbb{j} + \mathbb{k}$.

Definition 1. The higher order Leonardo quaternions $\left\{\mathcal{L}Q_n^{(k)}\right\}$, $n \in \mathbb{R}$, is defined as

$$\mathcal{L}Q_n^{(k)} = \mathcal{L}_n^{(k)} + \mathcal{L}_{n+1}^{(k)}\mathbb{i} + \mathcal{L}_{n+2}^{(k)}\mathbb{j} + \mathcal{L}_{n+3}^{(k)}\mathbb{k} \quad (7)$$

where $\mathbb{i}, \mathbb{j}, \mathbb{k}$ are quaternionic units and $\mathcal{L}_n^{(k)}$ is nth the higher order Leonardo numbers.

If it is taken as $k = 1$, the higher-order Leonardo quaternions $\mathcal{L}Q_n^{(1)}$ is called as the Leonardo quaternions.

The real and imaginary parts of the higher-order Leonardo quaternions are as follows:

$$\operatorname{Re}\left(\mathcal{L}Q_n^{(k)}\right) = \mathcal{L}_n^{(k)},$$

$$\operatorname{Im}\left(\mathcal{L}Q_n^{(k)}\right) = \mathcal{L}_{n+1}^{(k)}\mathbb{i} + \mathcal{L}_{n+2}^{(k)}\mathbb{j} + \mathcal{L}_{n+3}^{(k)}\mathbb{k}$$

, respectively.

The conjugate of the higher-order Leonardo quaternion, denoted by $\overline{\mathcal{L}Q_n^{(k)}}$, is given by $\overline{\mathcal{L}Q_n^{(k)}} = \mathcal{L}_n^{(k)} - \mathcal{L}_{n+1}^{(k)}\mathbb{i} - \mathcal{L}_{n+2}^{(k)}\mathbb{j} - \mathcal{L}_{n+3}^{(k)}\mathbb{k}$.

The norm of the higher-order Leonardo quaternion, denoted by $N(\mathcal{L}Q_n^{(k)})$, is given by

$$N\left(\mathcal{L}Q_n^{(k)}\right) = \mathcal{L}Q_n^{(k)}\overline{\mathcal{L}Q_n^{(k)}}$$
$$= (\mathcal{L}_n^{(k)})^2 + (\mathcal{L}_{n+1}^{(k)})^2 + (\mathcal{L}_{n+2}^{(k)})^2 + (\mathcal{L}_{n+3}^{(k)})^2.$$

Proposition 2. The higher-order Leonardo quaternions satisfy the following identity:

$$(\mathcal{L}Q_n^{(k)})^2 = -\mathcal{L}Q_n^{(k)}\overline{\mathcal{L}Q_n^{(k)}} + 2\mathcal{L}_n^{(k)}\mathcal{L}Q_n^{(k)}.$$

Proof. From the definition of $\mathcal{L}Q_n^{(k)}$, we have

$$\left(\mathcal{L}Q_n^{(k)}\right)^2 = \left(\mathcal{L}_n^{(k)} + \mathcal{L}_{n+1}^{(k)}\mathbb{i} + \mathcal{L}_{n+2}^{(k)}\mathbb{j} + \mathcal{L}_{n+3}^{(k)}\mathbb{k}\right)\left(\mathcal{L}_n^{(k)} + \mathcal{L}_{n+1}^{(k)}\mathbb{i} + \mathcal{L}_{n+2}^{(k)}\mathbb{j} + \mathcal{L}_{n+3}^{(k)}\mathbb{k}\right)$$

$$= (\mathcal{L}_n^{(k)})^2 - (\mathcal{L}_{n+1}^{(k)})^2 - (\mathcal{L}_{n+2}^{(k)})^2 - (\mathcal{L}_{n+3}^{(k)})^2 + 2\mathcal{L}_n^{(k)}\left(\mathcal{L}_{n+1}^{(k)}\mathbb{i} + \mathcal{L}_{n+2}^{(k)}\mathbb{j} + \mathcal{L}_{n+3}^{(k)}\mathbb{k}\right)$$

$$= (\mathcal{L}_n^{(k)})^2 - (\mathcal{L}_{n+1}^{(k)})^2 - (\mathcal{L}_{n+2}^{(k)})^2 - (\mathcal{L}_{n+3}^{(k)})^2 + 2\mathcal{L}_n^{(k)}\left(\mathcal{L}_{n+1}^{(k)}\mathbb{i} + \mathcal{L}_{n+2}^{(k)}\mathbb{j} + \mathcal{L}_{n+3}^{(k)}\mathbb{k}\right)$$

$$= -(\mathcal{L}_n^{(k)})^2 - (\mathcal{L}_{n+1}^{(k)})^2 - (\mathcal{L}_{n+2}^{(k)})^2 - (\mathcal{L}_{n+3}^{(k)})^2 + 2\mathcal{L}_n^{(k)}\left(\mathcal{L}_n^{(k)} + \mathcal{L}_{n+1}^{(k)}\mathbb{i} + \mathcal{L}_{n+2}^{(k)}\mathbb{j} + \mathcal{L}_{n+3}^{(k)}\mathbb{k}\right)$$

$$= -\mathcal{L}Q_n^{(k)}\overline{\mathcal{L}Q_n^{(k)}} + 2\mathcal{L}_n^{(k)}\mathcal{L}Q_n^{(k)}.$$

Theorem 3. The Binet's formula of the higher order Leonardo quaternions $\mathcal{L}_n^{(k)}$ is given by

$$\mathcal{L}Q_n^{(k)} = \frac{2\alpha^{kn+1}\hat{\alpha} - 2\beta^{kn+1}\hat{\beta} + w(-\alpha+\beta)}{2\alpha^{k+1} - 2\beta^{k+1} - \alpha + \beta} \tag{8}$$

where $\hat{\alpha} = 1 + \alpha^k\mathbb{i} + \alpha^{2k}\mathbb{j} + \alpha^{3k}\mathbb{k}$, $\hat{\beta} = 1 + \beta^k\mathbb{i} + \beta^{2k}\mathbb{j} + \beta^{3k}\mathbb{k}$ and $w = 1 + \mathbb{i} + \mathbb{j} + \mathbb{k}$.

Proof. Using the equations (6) and (7), we obtain

$$\mathcal{L}Q_n^{(k)} = \mathcal{L}_n^{(k)} + \mathcal{L}_{n+1}^{(k)}\mathbb{i} + \mathcal{L}_{n+2}^{(k)}\mathbb{j} + \mathcal{L}_{n+3}^{(k)}\mathbb{k}$$

$$= \frac{2\alpha^{kn+1}-2\beta^{kn+1}-\alpha+\beta}{2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta} + \frac{2\alpha^{kn+k+1}-2\beta^{kn+1}-\alpha+\beta}{2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta}\mathbb{i} + \frac{2\alpha^{kn+2k+1}-2\beta^{kn+2k+1}-\alpha+\beta}{2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta}\mathbb{j} + \frac{2\alpha^{kn+3k+1}-2\beta^{kn+3k+1}-\alpha+\beta}{2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta}\mathbb{k}$$

$$= \frac{1}{2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta}(2\alpha^{kn+1}(1+\alpha^k\mathbb{i}+\alpha^{2k}\mathbb{j}+\alpha^{3k}\mathbb{k}) - 2\beta^{kn+1}(1+\beta^k\mathbb{i}+\beta^{2k}\mathbb{j}+\beta^{3k}\mathbb{k}) - \alpha + \beta(1+\mathbb{i}+\mathbb{j}+\mathbb{k}))$$

$$= \frac{2\alpha^{kn+1}\widehat{\alpha}-2\beta^{kn+1}\widehat{\beta}+w(-\alpha+\beta)}{2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta}.$$

Theorem 4. For the higher order Leonardo quaternions, the following relation holds:

$$\mathcal{L}Q_{n+1}^{(k)} = \frac{1}{2}(\mathcal{L}_k\mathcal{L}Q_n^{(k)} - (-1)^k\mathcal{L}Q_{n-1}^{(k)} + \frac{1}{5}\mathcal{L}_k^{-1}[-4K_{kn+k} + 2(-1)^{k+1}K_{kn-k+1}] + \mathcal{L}Q_n^{(k)} + w(1-\mathcal{L}_k^{-1}(3-(-1)^k))).$$

where K_n is the nth Lucas quaternion.

Proof. From the Binet formulas (1) and (8), we obtain as follows:

$$\mathcal{L}_k\mathcal{L}Q_n^{(k)} - (-1)^k\mathcal{L}Q_{n-1}^{(k)} = \frac{2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta}{\alpha-\beta}\frac{2\alpha^{kn+1}\widehat{\alpha}-2\beta^{kn+1}\widehat{\beta}+w(-\alpha+\beta)}{2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta} - (\alpha\beta)^k\frac{2\alpha^{kn-k+1}\widehat{\alpha}-2\beta^{kn-k+1}\widehat{\beta}+w(-\alpha+\beta)}{2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta}$$

$$= \frac{1}{(2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta)(\alpha-\beta)}\Big(\hat{\alpha}(4\alpha^{kn+k+2} - 4\alpha^{kn+1}\beta^{k+1} - 2\alpha^{kn+2} + 2\beta\alpha^{kn+1} - 2\beta^k\alpha^{kn+2} + 2\beta^{k+1}\alpha^{kn+1}) + \hat{\beta}(4\beta^{kn+k+2} - 4\alpha^{k+1}\beta^{kn+1} - 2\beta^{kn+2} + 2\alpha\beta^{kn+1} + 2\beta^{kn+1}\alpha^{k+1} - 2\beta^{kn+2}\alpha^k) + w(-\alpha+\beta)(2\alpha^{k+1} - 2\beta^{k+1} - \alpha + \beta - \alpha^{k+1}\beta^k + \alpha^k\beta^{k+1})\Big)$$

$$= \frac{1}{(2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta)(\alpha-\beta)}\Big(2(\alpha-\beta)\big(2\hat{\alpha}\alpha^{kn+k+1} - 2\hat{\beta}\beta^{kn+k+1}\big) + 4\hat{\beta}\alpha\beta^{kn+k+1} + 4\hat{\alpha}\beta\alpha^{kn+k+1} -$$

$2\hat{\alpha}(\alpha\beta)^{k+1}\alpha^{kn-k} + 2\hat{\alpha}(\alpha\beta)^{k+1}\alpha^{kn-k+2} - 2\hat{\alpha}\alpha^{kn+1}(\alpha-\beta) -$
$2\hat{\beta}(\alpha\beta)^{k+1}\beta^{kn-k} + 2\hat{\beta}(\alpha\beta)^{k+1}\beta^{kn-k+2} - 2\hat{\beta}\beta^{kn+1}(-\alpha+\beta) +$
$w(-\alpha+\beta)\Big(2\alpha^{k+1} - 2\beta^{k+1} - \alpha + \beta - (\alpha\beta)^k(\alpha-\beta)\Big)\Big)$

$= \frac{1}{(2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta)(\alpha-\beta)}\Big(2(\alpha-\beta)\Big(2\hat{\alpha}\alpha^{kn+k+1} -$
$2\hat{\beta}\beta^{kn+k+1} + w(-\alpha+\beta)\Big) - 4(\hat{\beta}\beta^{kn+k} + \hat{\alpha}\alpha^{kn+k}) -$
$2(\alpha\beta)^{k+1}\big(\hat{\alpha}\alpha^{kn-k} + \hat{\beta}\beta^{kn-k}\big) + 2(\alpha\beta)^{k+1}(\hat{\alpha}\alpha^{kn-k+2} +$
$\hat{\beta}\beta^{kn-k+2}) - 2(\alpha-\beta)(\hat{\alpha}\alpha^{kn+1} - \hat{\beta}\beta^{kn+1}) + w(-\alpha +$
$\beta)\Big(2\alpha^{k+1} - 2\beta^{k+1} - 3\alpha + 3\beta - (\alpha\beta)^k(\alpha-\beta)\Big)\Big)$
$= 2\,\frac{2\hat{\alpha}\alpha^{kn+k+1} - 2\hat{\beta}\beta^{kn+k+1} + w(-\alpha+\beta)}{(2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta)} -$
$\frac{1}{(2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta)}\Big[\frac{-4(\hat{\beta}\beta^{kn+k}+\hat{\alpha}\alpha^{kn+k})}{(\alpha-\beta)} -$
$2(\alpha\beta)^{k+1}\frac{(\hat{\alpha}\alpha^{kn-k}+\hat{\beta}\beta^{kn-k})}{(\alpha-\beta)} + 2(\alpha\beta)^{k+1}\frac{\hat{\alpha}\alpha^{kn-k+2}+\hat{\beta}\beta^{kn-k+2}}{\alpha-\beta}\Big] -$
$\frac{1}{(2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta)(\alpha-\beta)}\Big[(\alpha-\beta)\Big(2\hat{\alpha}\alpha^{kn+1} - 2\hat{\beta}\beta^{kn+1} +$
$w(-\alpha+\beta)\Big) + w(-\alpha+\beta)(2\alpha^{k+1} - 2\beta^{k+1} - 4\alpha + 4\beta -$
$(\alpha\beta)^k(\alpha-\beta))\Big]$
$= 2\mathcal{L}Q_{n+1}^{(k)} - \frac{1}{5}\mathcal{L}_k^{-1}[-4K_{kn+k} - 2(-1)^{k+1}(K_{kn-k} - K_{kn-k+2})] -$
$\mathcal{L}Q_n^{(k)} - w(1 - \mathcal{L}_k^{-1}(3 - (-1)^k)).$

So, we have

$\mathcal{L}Q_{n+1}^{(k)} = \frac{1}{2}(\mathcal{L}_k\mathcal{L}Q_n^{(k)} - (-1)^k\mathcal{L}Q_{n-1}^{(k)} + \frac{1}{5}\mathcal{L}_k^{-1}[-4K_{kn+k} +$
$2(-1)^{k+1}K_{kn-k+1}] + \mathcal{L}Q_n^{(k)} + w(1 - \mathcal{L}_k^{-1}(3 - (-1)^k))).$

Theorem 5. The generating function of the higher-order Leonardo quaternions is given by

$$G(x,k) = \frac{\mathcal{L}_k^{-1}(\mathcal{L}Q_0 + (-\mathcal{L}Q_0 - \mathcal{L}Q_{k-2} + w(L_k - 2))x) + (-\mathcal{L}Q_{k-2} - w((-1)^k + 1))x^2))}{1 - (1 + L_k)x - (L_k + (-1)^k)x^2 - (-1)^k x^3}.$$

Proof. Suppose that the generating function for the the higher-order Leonardo quaternions is

$$G(x,k)=\sum_{n=0}^{\infty}\mathcal{L}Q_n^{(k)}x^n=\mathcal{L}Q_0^{(k)}+\mathcal{L}Q_1^{(k)}x+\mathcal{L}Q_2^{(k)}x^2+\cdots+\mathcal{L}Q_n^{(k)}x^n+\cdots.$$

By using Binet's formula of $\mathcal{L}_n^{(k)}$ (5), we can rewrite the generating function as follows:

$$G(x,k)=\sum_{n=0}^{\infty}\mathcal{L}Q_n^{(k)}x^n=\sum_{n=0}^{\infty}\left(\mathcal{L}_n^{(k)}+\mathcal{L}_{n+1}^{(k)}\mathbb{i}+\mathcal{L}_{n+2}^{(k)}\mathbb{j}+\mathcal{L}_{n+3}^{(k)}\mathbb{k}\right)x^n$$

$$=\sum_{n=0}^{\infty}\left(\frac{2\alpha^{kn+1}-2\beta^{kn+1}-\alpha+\beta}{2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta}+\frac{2\alpha^{kn+k+1}-2\beta^{kn+k+1}-\alpha+\beta}{2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta}\mathbb{i}+\frac{2\alpha^{kn+2k+1}-2\beta^{kn+2k+1}-\alpha+\beta}{2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta}\mathbb{j}+\frac{2\alpha^{kn+3k+1}-2\beta^{kn+3k+1}-\alpha+\beta}{2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta}\mathbb{k}\right)$$

$$=\frac{1}{2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta}\left(2\sum_{n=0}^{\infty}\alpha^{kn+1}(1+\alpha^k\mathbb{i}+\alpha^{2k}\mathbb{j}+\alpha^{3k}\mathbb{k})x^n-2\sum_{n=0}^{\infty}\beta^{kn+1}(1+\beta^k\mathbb{i}+\beta^{2k}\mathbb{j}+\beta^{3k}\mathbb{k})x^n+(-\alpha+\beta)\sum_{n=0}^{\infty}(1+\mathbb{i}+\mathbb{j}+\mathbb{k})x^n\right)$$

$$=\frac{1}{2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta}\left(2\alpha\hat{\alpha}\sum_{n=0}^{\infty}(\alpha^kx)^n-2\beta\hat{\beta}\sum_{n=0}^{\infty}(\beta^kx)^n+(-\alpha+\beta)w\sum_{n=0}^{\infty}x^n\right)$$

$$=\frac{1}{2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta}\left(\frac{2\alpha\hat{\alpha}}{1-\alpha^kx}-\frac{2\beta\hat{\beta}}{1-\beta^kx}+(-\alpha+\beta)w\frac{1}{1-x}\right)$$

$$=\frac{1}{2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta}\frac{\left(2\alpha\hat{\alpha}(1-\beta^kx)(1-x)-2\beta\hat{\beta}(1-\alpha^kx)(1-x)+(-\alpha+\beta)w(1-\alpha^kx)(1-\beta^kx)\right)}{1-(1+\alpha^k+\beta^k)x-\left(\alpha^k+\beta^k+(\alpha\beta)^k\right)x^2-(\alpha\beta)^kx^3}$$

$$=\frac{1}{2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta}\frac{2\alpha\hat{\alpha}-2\beta\hat{\beta}+(-\alpha+\beta)w+\left(-2\alpha\hat{\alpha}-2\hat{\alpha}\alpha\beta^k+2\hat{\beta}\beta+2\hat{\beta}\beta\alpha^k+w(-\alpha+\beta)(-\alpha^k-\beta^k)\right)x}{1-(1+\alpha^k+\beta^k)x-\left(\alpha^k+\beta^k+(\alpha\beta)^k\right)x^2-(\alpha\beta)^kx^3}$$

$$+\frac{1}{2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta}\frac{\left(2\hat{\alpha}\alpha\beta^k-2\hat{\beta}\beta\alpha^k+(-\alpha+\beta)w\alpha^k\beta^k\right)x^2}{1-(1+\alpha^k+\beta^k)x-\left(\alpha^k+\beta^k+(\alpha\beta)^k\right)x^2-(\alpha\beta)^kx^3}$$

$$=\frac{\mathcal{L}_k^{-1}\left(\mathcal{L}Q_0+(-\mathcal{L}Q_0+(-1)^{k+1}\mathcal{L}Q_{-k+1}+w(L_k+(-1)^{k+1}-1))x+((-1)^k\mathcal{L}Q_{-k+1}+w((-1)^k-(-1)^k))x^2\right)}{1-(1+L_k)x-\left(L_k+(-1)^k\right)x^2-(-1)^kx^3}$$

$$= \frac{\mathcal{L}_k^{-1}\left(\mathcal{L}Q_0+(-\mathcal{L}Q_0+(-1)^{k+1}\mathcal{L}Q_{-k+1}+w(L_k+(-1)^{k+1}-1))x+(-1)^k\mathcal{L}Q_{-k+1}x^2\right)}{1-(1+L_k)x-\left(L_k+(-1)^k\right)x^2-(-1)^k x^3}.$$

Theorem 6. The sum of the higher-order Leonardo quaternions is given by

$$\sum_{n=0}^{\infty} \mathcal{L}Q_n^{(k)} = \frac{w\mathcal{L}_k^{-1}(L_k-(-1)^k-1)}{-2(L_k+(-1)^k)}.$$

Proof. If we take for $x = 1$ in Theorem 5, the proof is completed.

Lemma 7. Let $\hat{\alpha} = 1 + \alpha^k \mathbb{i} + \alpha^{2k}\mathbb{j} + \alpha^{3k}\mathbb{k}$, $\hat{\beta} = 1 + \beta^k \mathbb{i} + \beta^{2k}\mathbb{j} + \beta^{3k}\mathbb{k}$ and $w = 1 + \mathbb{i} + \mathbb{j} + \mathbb{k}$. Then, there are the following equations

$$\hat{\alpha}\hat{\beta} = u - lv \tag{9}$$

$$\hat{\beta}\hat{\alpha} = u + lv \tag{10}$$

where $u = -2(-1)^k + L_k\mathbb{i} + L_{2k}\mathbb{j} + L_{3k}\mathbb{k}$, $v = \mathbb{i} - (-1)^k L_k \mathbb{j} + (-1)^k\mathbb{k}$ and $l = \alpha^k - \beta^k$.

Proof. By using the equations (2) and (3), we obtain as follows:

$\hat{\alpha}\hat{\beta} = (1 + \alpha^k\mathbb{i} + \alpha^{2k}\mathbb{j} + \alpha^{3k}\mathbb{k})(1 + \beta^k\mathbb{i} + \beta^{2k}\mathbb{j} + \beta^{3k}\mathbb{k})$

$= 1 + \beta^k\mathbb{i} + \beta^{2k}\mathbb{j} + \beta^{3k}\mathbb{k} + \alpha^k\mathbb{i} - (\alpha\beta)^k + \alpha^k\beta^{2k}\mathbb{i}\mathbb{j} + \alpha^k\beta^{3k}\mathbb{i}\mathbb{k} + \alpha^{2k}\mathbb{j} + \alpha^{2k}\beta^k\mathbb{j}\mathbb{i} - \alpha^{2k}\beta^{2k} + \alpha^{2k}\beta^{3k}\mathbb{j}\mathbb{k} + \alpha^{3k}\mathbb{k} + \alpha^{3k}\beta^k\mathbb{k}\mathbb{i} + \alpha^{3k}\beta^{2k}\mathbb{k}\mathbb{j} - \alpha^{3k}\beta^{3k}$

$= 1 - (-1)^k - (-1)^{2k} - (-1)^{3k} + (\beta^k + \alpha^k + \alpha^{2k}\beta^{3k} - \alpha^{3k}\beta^{2k})\mathbb{i} + (\alpha^{2k} + \beta^{2k} - \alpha^k\beta^{3k} + \alpha^{3k}\beta^k)\mathbb{j} + (\alpha^{3k} + \beta^{3k} + \alpha^k\beta^{2k} - \alpha^{2k}\beta^k)\mathbb{k}$

$= -(-1)^k - (-1)^{3k} + (\beta^k + \alpha^k - \alpha^{2k}\beta^{2k}(\alpha^k - \beta^k))\mathbb{i} + (\alpha^{2k} + \beta^{2k} + \alpha^k\beta^k(\alpha^{2k}-\beta^{2k}))\mathbb{j} + (\alpha^{3k} + \beta^{3k} - \alpha^k\beta^k(\alpha^k - \beta^k))\mathbb{k}$

$$= -2(-1)^k + (\alpha^k + \beta^k)\mathbb{i} + (\alpha^{2k} + \beta^{2k})\mathbb{j} + (\alpha^{3k} + \beta^{3k})\mathbb{k} - (\alpha^{2k}\beta^{2k}(\alpha^k - \beta^k)\mathbb{i} - \alpha^k\beta^k(\alpha^{2k} - \beta^{2k})\mathbb{j} + \alpha^k\beta^k(\alpha^k - \beta^k)\mathbb{k})$$

$$= -2(-1)^k + L_k\mathbb{i} + L_{2k}\mathbb{j} + L_{3k}\mathbb{k} - (\alpha^k - \beta^k)(\mathbb{i} - (-1)^k L_k\mathbb{j} + (-1)^k\mathbb{k}))$$

$$= u - lv$$.

$$\hat{\beta}\hat{\alpha} = (1 + \beta^k\mathbb{i} + \beta^{2k}\mathbb{j} + \beta^{3k}\mathbb{k})(1 + \alpha^k\mathbb{i} + \alpha^{2k}\mathbb{j} + \alpha^{3k}\mathbb{k})$$

$$= 1 + \alpha^k\mathbb{i} + \alpha^{2k}\mathbb{j} + \alpha^{3k}\mathbb{k} + \beta^k\mathbb{i} - (\alpha\beta)^k + \alpha^{2k}\beta^k\mathbb{i}\mathbb{j} + \alpha^{3k}\beta^k\mathbb{i}\mathbb{k} + \beta\mathbb{j}^{2k} + \alpha^k\beta^{2k}\mathbb{j}\mathbb{i} - \alpha^{2k}\beta^{2k} + \alpha^{3k}\beta^{2k}\mathbb{j}\mathbb{k} + \beta^{3k}\mathbb{k} + \alpha^k\beta^{3k}\mathbb{k}\mathbb{i} + \alpha^{2k}\beta^{3k}\mathbb{k}\mathbb{j} - \alpha^{3k}\beta^{3k}$$

$$= 1 - (-1)^k - (-1)^{2k} - (-1)^{3k} + (\alpha^k + \beta^k + (-1)^{2k}(\alpha^k - \beta^k))\mathbb{i} + (\alpha^{2k} + \beta^{2k} - (-1)^k(\alpha^{2k} - \beta^{2k})\mathbb{j} + (\alpha^{3k} + \beta^{3k} + \alpha^k\beta^k(\alpha^k - \beta^k))\mathbb{k}$$

$$= -2(-1)^k + (\alpha^k + \beta^k)\mathbb{i} + (\alpha^{2k} + \beta^{2k})\mathbb{j} + (\alpha^{3k} + \beta^{3k})\mathbb{k} + ((-1)^{2k}(\alpha^k - \beta^k)\mathbb{i} - (-1)^k(\alpha^{2k} - \beta^{2k})\mathbb{j} + (-1)^k(\alpha^k - \beta^k)\mathbb{k})$$

$$= -2(-1)^k + L_k\mathbb{i} + L_{2k}\mathbb{j} + L_{3k}\mathbb{k} + (\alpha^k - \beta^k)((\mathbb{i} - (-1)^k L_k\mathbb{j} + (-1)^k\mathbb{k}))$$

$$= u + lv.$$

Theorem 8. For any $n, m, r \in Z$, Vajda identity for the higher order Leonardo quaternions is given by

$$\mathcal{L}Q_{n+m}^{(k)}\mathcal{L}Q_{n+r}^{(k)} - \mathcal{L}Q_n^{(k)}\mathcal{L}Q_{n+m+r}^{(k)} = \mathcal{L}_k^{-2}(4(-1)^{kn+1}F_{km}(uF_{kr} + vF_{kr}L_{kr}) - w(\mathcal{L}Q_{kn+kr} - \mathcal{L}Q_{kn+km+kr}) + (\mathcal{L}Q_{kn+km} - \mathcal{L}Q_{kn})w).$$

Proof. Left side of the equation is rewritten with the help of the Binet formula of $\mathcal{L}Q_n^{(k)}$ as follows:

$$\mathcal{L}Q_{n+m}^{(k)}\mathcal{L}Q_{n+r}^{(k)} = \frac{2\hat{\alpha}\alpha^{kn+km+1}-2\hat{\beta}\beta^{kn+km+1}+w(-\alpha+\beta)}{2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta} \cdot \frac{2\hat{\alpha}\alpha^{kn+kr+1}-2\hat{\beta}\beta^{kn+kr+1}+w(-\alpha+\beta)}{2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta}$$
$$= \frac{1}{\left(2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta\right)^2}\Big(4\hat{\alpha}\hat{\alpha}\alpha^{2kn+km+kr+2} + 4\hat{\beta}\hat{\beta}\beta^{2kn+km+kr+2} - 4\hat{\alpha}\hat{\beta}\alpha^{kn+km+1}\beta^{kn+kr+1} - 4\hat{\beta}\hat{\alpha}\beta^{kn+km+1}\alpha^{kn+kr+1} + 2\hat{\alpha}w\alpha^{kn+km+1}(-\alpha+\beta) - 2\hat{\beta}w\beta^{kn+km+1}(-\alpha+\beta) + w(-\alpha+\beta)\big(2\hat{\alpha}\alpha^{kn+kr+1} - 2\hat{\beta}\beta^{kn+kr+1}\big) + w^2(-\alpha+\beta)^2\Big). \quad (11)$$

$$\mathcal{L}Q_{n}^{(k)}\mathcal{L}Q_{n+m+r}^{(k)} = \frac{2\hat{\alpha}\alpha^{kn+1}-2\hat{\beta}\beta^{kn+1}+w(-\alpha+\beta)}{2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta} \cdot \frac{2\hat{\alpha}\alpha^{kn+km+kr+1}-2\hat{\beta}\beta^{kn+km+kr+1}+w(-\alpha+\beta)}{2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta}$$
$$= \frac{1}{\left(2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta\right)^2}\Big(4\hat{\alpha}\hat{\alpha}\alpha^{2kn+km+kr+2} + 4\hat{\beta}\hat{\beta}\beta^{2kn+km+kr+2} - 4\hat{\alpha}\hat{\beta}\alpha^{kn+1}\beta^{kn+km+kr+1} - 4\hat{\beta}\hat{\alpha}\beta^{kn+1}\alpha^{kn+km+kr+1} + 2\hat{\alpha}w\alpha^{kn+1}(-\alpha+\beta) - 2\hat{\beta}w\beta^{kn+1}(-\alpha+\beta) + w(-\alpha+\beta)\big(2\hat{\alpha}\alpha^{kn+km+kr+1} - 2\hat{\beta}\beta^{kn+km+kr+1}\big) + w^2(-\alpha+\beta)^2\Big). \quad (12)$$

Substracting the equation (11) from the equation (12), we obtain

$$\mathcal{L}Q_{n+m}^{(k)}\mathcal{L}Q_{n+r}^{(k)} - \mathcal{L}Q_{n}^{(k)}\mathcal{L}Q_{n+m+r}^{(k)} = \frac{1}{\left(2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta\right)^2} - 4(\alpha\beta)^{kn+1}\Big(\hat{\alpha}\hat{\beta}\beta^{kr}(\alpha^{km}-\beta^{km}) - \hat{\beta}\hat{\alpha}\alpha^{kr}(\alpha^{km}-\beta^{km})\Big) + 2w(-\alpha+\beta)\Big(\big(\hat{\alpha}\alpha^{kn+kr+1} - \hat{\beta}\beta^{kn+kr+1}\big) - \big(\hat{\alpha}\alpha^{kn+km+kr+1} - \hat{\beta}\beta^{kn+km+kr+1}\big)\Big) + 2(-\alpha+\beta)[\big(\hat{\alpha}\alpha^{kn+km+1} - 2\hat{\beta}\beta^{kn+km+1}\big)w - \big(2\hat{\alpha}\alpha^{kn+1} - 2\hat{\beta}\beta^{kn+1}\big)w]$$

$$= \frac{1}{\left(2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta\right)^2}\Big(-4(-1)^{kn+1}(\alpha^{km}-\beta^{km})\big(\hat{\alpha}\hat{\beta}\beta^{kr} - \hat{\beta}\hat{\alpha}\alpha^{kr}\big) + 2(-\alpha+\beta)[w\Big(\big(\hat{\alpha}\alpha^{kn+kr+1} - \hat{\beta}\beta^{kn+kr+1}\big) - \big(\hat{\alpha}\alpha^{kn+km+kr+1} - \hat{\beta}\beta^{kn+km+kr+1}\big)\Big) + \big(\hat{\alpha}\alpha^{kn+km+1} - 2\hat{\beta}\beta^{kn+km+1}\big)w - \big(2\hat{\alpha}\alpha^{kn+1} - 2\hat{\beta}\beta^{kn+1}\big)w]\Big)$$

$$= \frac{(\alpha-\beta)^2}{\left(2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta\right)^2}\Big(-4(-1)^{kn+1}\frac{(\alpha^{km}-\beta^{km})}{(\alpha-\beta)^2}(u\beta^{kr} - lv\beta^{kr} - u\alpha^{kr} - lv\alpha^{kr}) + 2(-\alpha+\beta)\frac{1}{(\alpha-\beta)^2}[w\Big(\big(\hat{\alpha}\alpha^{kn+kr+1} - \hat{\beta}\beta^{kn+kr+1} - \alpha + \beta\big) - \big(\hat{\alpha}\alpha^{kn+km+kr+1} - \hat{\beta}\beta^{kn+km+kr+1} - \alpha + \beta\big)\Big) + \big(\hat{\alpha}\alpha^{kn+km+1} - 2\hat{\beta}\beta^{kn+km+1} - \alpha + \beta\big)w - \big(2\hat{\alpha}\alpha^{kn+1} - 2\hat{\beta}\beta^{kn+1} - \alpha + \beta\big)w]\Big)$$

$$= \frac{(\alpha-\beta)^2}{\left(2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta\right)^2}\Big(-4(-1)^{kn+1}\frac{(\alpha^{km}-\beta^{km})}{(\alpha-\beta)^2}(-u(\alpha^{kr} - \beta^{kr}) - lv(\alpha^{kr} + \beta^{kr})) + 2(-\alpha+\beta)\frac{1}{(\alpha-\beta)^2}[w\Big(\big(\hat{\alpha}\alpha^{kn+kr+1} - \hat{\beta}\beta^{kn+kr+1} + w(-\alpha + \beta)\big) - \big(\hat{\alpha}\alpha^{kn+km+kr+1} - \hat{\beta}\beta^{kn+km+kr+1} + w(-\alpha + \beta)\big)\Big) + \big(\hat{\alpha}\alpha^{kn+km+1} - 2\hat{\beta}\beta^{kn+km+1} + w(-\alpha + \beta)\big)w - \big(2\hat{\alpha}\alpha^{kn+1} - 2\hat{\beta}\beta^{kn+1} + w(-\alpha + \beta)\big)w]\Big).$$

By using the equations (1), (6), (9) and (10), we have

$$\mathcal{L}Q_{n+m}^{(k)}\mathcal{L}Q_{n+r}^{(k)} - \mathcal{L}Q_{n}^{(k)}\mathcal{L}Q_{n+m+r}^{(k)} = \mathcal{L}_k^{-2}(4(-1)^{kn+1}F_{km}(uF_{kr} + vF_{kr}L_{kr}) - w(\mathcal{L}Q_{kn+kr} - \mathcal{L}Q_{kn+km+kr}) + (\mathcal{L}Q_{kn+km} - \mathcal{L}Q_{kn})w).$$

Theorem 9. For any $r \leq n \in \mathbb{Z}$, Catalan's identity for the higher order Leonardo quaternions is

$$\mathcal{L}Q_{n-r}^{(k)}\mathcal{L}Q_{n+r}^{(k)} - \left(\mathcal{L}Q_{n}^{(k)}\right)^2 = \mathcal{L}_k^{-2}(4(-1)^{kn+1}F_{-kr}(uF_{kr} + vF_{kr}L_{kr}) - w(\mathcal{L}Q_{kn+kr} - \mathcal{L}Q_{kn}) + (\mathcal{L}Q_{kn-kr} - \mathcal{L}Q_{kn})w).$$

Proof. The proof of Catalan's identity is done for the special case $m = -r$ of Vajda identity.

Theorem 10. Cassini's identity for the higher order Leonardo quaternions is

$$\mathcal{L}Q_{n-1}^{(k)}\mathcal{L}Q_{n+1}^{(k)} - \left(\mathcal{L}Q_{n}^{(k)}\right)^2 = \mathcal{L}_k^{-2}(4(-1)^{kn+1}F_{-k}(uF_{k} + vF_{k}L_{k}) - w(\mathcal{L}Q_{kn+k} - \mathcal{L}Q_{kn}) + (\mathcal{L}Q_{kn-k} - \mathcal{L}Q_{kn})w).$$

Proof. When it is taken as $r = 1$ and $m = -1$ in Vajda identity, we have

$$\mathcal{L}Q_{n-1}^{(k)}\mathcal{L}Q_{n+1}^{(k)} - \left(\mathcal{L}Q_n^{(k)}\right)^2 = \mathcal{L}_k^{-2}(4(-1)^{kn+1}F_{-k}(uF_k + vF_kL_k) - w(\mathcal{L}Q_{kn+k} - \mathcal{L}Q_{kn}) + (\mathcal{L}Q_{kn-k} - \mathcal{L}Q_{kn})w).$$

Theorem 11. D'Ocagne's identity for the higher-order Leonardo quaternions is

$$\mathcal{L}Q_k^{(k)}\mathcal{L}Q_{n+1}^{(k)} - \mathcal{L}Q_n^{(k)}\mathcal{L}Q_{k+1}^{(k)} = \mathcal{L}_k^{-2}(4(-1)^{kn+1}F_{k(k-n)}(uF_k + vF_kL_k) - w\left(\mathcal{L}Q_{k(n+1)} - \mathcal{L}Q_{k(k+1)}\right) + (\mathcal{L}Q_{k^2} - \mathcal{L}Q_{kn})w).$$

Proof. If we take as $m + n = k$ and $r = 1$ in Vajda identity, we obtain as follows:

$$\mathcal{L}Q_k^{(k)}\mathcal{L}Q_{n+1}^{(k)} - \mathcal{L}Q_n^{(k)}\mathcal{L}Q_{k+1}^{(k)} = \mathcal{L}_k^{-2}(4(-1)^{kn+1}F_{k(k-n)}(uF_k + vF_kL_k) - w\left(\mathcal{L}Q_{k(n+1)} - \mathcal{L}Q_{k(k+1)}\right) + (\mathcal{L}Q_{k^2} - \mathcal{L}Q_{kn})w).$$

Conclusion

In this paper, we have defined the higher order Leonardo quaternions with the higher order Leonardo number components. We have introduced basic definitions and properties of these quaternions. Then, we have obtained fundamental properties such as Binet formula and generating function. Moreover, we have proved several identities such as Vajda identity, Catalan's identity, Cassini's identity and d'Ocagne identity for the higher order Leonardo quaternions by using Binet formula.

REFERENCES

Abrate, M., Barbero, S., Cerruti, U. & Murru, N. (2014). Colored compositions, invert operator andelegant compositions with the black tie. *Discrete Mathematics, 335,* 1-7. Doi: 10.1016/j.disc.2014.06.026

Alp, Y. & Koçer, E. G. (2021). Some Properties of Leonardo Numbers. *Konuralp Journal of Mathematics*, *9* (1), 183-189.

Asci M., Aydinyuz S., (2021). *k*-order Fibonacci quaternions, Journal of Science and Arts; Targoviste, 21(1), 29-38. Doi:10.46939/J.Sci.Arts-21.1-a04

Amannah, C. I. & Nanwin, N. D. (2014). The Existence of Fibonacci numbers in the algorithmic generator for combinatoric Pascal triangle. *British Journal of Science*, *11*(2), 62-84.

Catarino, P. M., & Borges, A. (2019). On Leonardo numbers. *Acta Mathematica Universitatis Comenianae*, *89*(1), 75-86.

Cook, C. K. & Bacon, M. R. (2013). Some identities for Jacobsthal and Jacobsthal-Lucas numbers satisfying higher order recurrence relations. *Annales mathematicae et informaticae*, 27–39.

Falcon, S., & Plaza, A. (2007). On the *k*-Fibonacci Numbers. *Chaos, Solitons and Fractals*, *32*(5), 1615- 1624. doi:10.1016/j.chaos.2006.09.022

Gül, K. (2022). Generalized *k* -Order Fibonacci Hybrid Quaternions. *Erzincan University Journal of Science and Technology*, *15* (2), 670-683. Doi: 10.18185/erzifbed.1132164

Gül, K. (2023). Higher order Leonardo numbers. *Cumhuriyet 10th International Conference on Applied Sciences*, 29 October 2023, Turkey, (pp. 82-87).

Gül, K. (2019). On bi-periodic Jacobsthal and Jacobsthal-Lucas quaternions. *Journal of Mathematics Research*, *11*(2), 44-52. Doi:10.5539/JMR.V11N2P44

Gül, K. (2020). Dual bicomplex Horadam quaternions. *Notes on Number Theory and Discrete Mathematics*, *26*(4), 187–205. Doi: 10.7546/nntdm.2020.26.4.187-205

Halici, S. (2012). On Fibonacci quaternions, *Advances in Applied Clifford Algebras*, *22*(2), 321-327. Doi: 10.1007/s00006-011-0317-1

Hamilton, W. R. (1866). *Elements of quaternions*. Longmans, Green and Co., London.

Horadam, A. F. (1963). Complex Fibonacci numbers and Fibonacci quaternions. *The American Mathematical Monthly*, 70(3), 289-291. Doi: 10.2307/2313129

Isbilir, Z. M. A., & Tosun, M. (2023). Pauli–Leonardo quaternions. *Notes on Number Theory and Discrete Mathematics*, *29* (1), 1–16. Doi: 10.7546/nntdm.2023.29.1.1-16

Kizilates, C. & Kone, T. (2021a). On higher order Fibonacci quaternions. *The Journal of Analysis*, *29*(4), 1071-1082. Doi: 10.1007/s41478-020-00295-1

Kizilates, C. & Kone, T. (2021b). On higher order Fibonacci hyper complex numbers. *Chaos, Solitons & Fractals*, *148*. Doi: 10.1016/j.chaos.2021.111044

Koshy, T. (2001). *Fibonacci and Fibonacci Lucas Numbers with Applications*. John Wiley and Sons Inc., NY. Doi:10.1002/9781118033067

Mangueira, M. D. S., Alves, F. R. V. & Catarino, P. M. M. C. (2022). Hybrid Quaternions of Leonardo. *Trends in Computational and Applied Mathematics*, *23*(1), 51-62. Doi: 10.5540/tcam.2022.023.01.00051

Nurkan, S. K. & Güven, I. A. (2023). Ordered Leonardo Quadruple Numbers. *Symmetry 15* (1) Article ID 149 15 pages. Doi:10.3390/sym15010149

Ozimamoğlu, H. (2023). On hyper complex numbers with higher order Pell numbers components. The Journal of Analysis, 1–15. Doi:10.1007/s41478-023-00579-2

Özkan, E. & Uysal, M. (2023). On Quaternions with Higher Order Jacobsthal Numbers Components. *Gazi University Journal of Science*, *36* (1), 336-347. Doi: 10.35378/gujs. 1002454

Prasad, K., Kumari, M., Mohanta, R., & Mahato, H. (2023). The sequence of higher order Mersenne numbers and associated binomial transforms. *arXiv preprint arXiv:2307.08073*, 2023. Doi:10.48550/arXiv.2307.08073

Randić, M., Morales, D. A. & Araujo, O. (1996). Higher-order Fibonacci numbers. *Journal of Mathematical Chemistry*, 20, 79–94. Doi:10.1007/BF01165157

Shannon, A. G., Deveci, Ö. & Erdağ, Ö. (2019). Generalized Fibonacci numbers and Bernoulli polynomials. *Notes Number Theory Discrete Mathematics*, *25*(1), 193-198. Doi: 10.7546/nntdm.2019.25.1.193-198

Yılmaz, Ç. Z., & Saçlı, G. Y. (2023). On Dual Quaternions with *k*-Generalized Leonardo Components. *Journal of New Theory*, 44, 31-42. Doi: 10.53570/jnt.1328605

CHAPTER VII

Uniqueness of Coupled Common Fixed Point under a Symmetric Contraction in Ordered GV-FMS

Manish JAIN[1]

1. Introduction

In 1965, Zadeh [1] innovated the notion of fuzzy sets that lead the beginning of a new era providing quick headways into different branches of mathematics and its areas of applications. Particularly, metric space has been fuzzified in several inequivalent ways resulting into different definitions of fuzzy metric space [2 - 6].

The fuzzy version of the most celebrated “Banach contraction principle" in fuzzy metric spaces in the sense of Kramosi and Michalek (in short, KM) was presented by Grabiec [7]. The results proved by Fang [8] improved, generalized and unified the works of Edelstein [9], Istratescu [10], Sehgal and Bharucha-Reid [11].

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George and Veeramani [6, 12] modified the concept of fuzzy metric space due to Kramosil and Michalek [5] and thereby obtained Hausdorff topology in fuzzy metric space. After wards, various authors established several fixed point results in complete fuzzy metric spaces in the sense of George and Veeramani (GV) [6, 12], one can see ([13] – [14]). In the present work, we consider the definition of the fuzzy metric space introduced by George and Veeramani [6].

The work of Bhaskar and Lakshmikantham [15] is worth mentioning, as they introduced the new notion of fixed points for the mappings having domain the product space $X \times X$, which they called coupled fixed points, and thereby proved some coupled fixed point theorems for mappings satisfying the mixed monotone property in partially ordered metric spaces. As an application, they discussed the existence and uniqueness of a solution for a periodic boundary value problem. Lakshmikantham and Ćirić [16] extended the notion of the mixed monotone property to the mixed g-monotone property and generalized the results of Bhaskar and Lakshmikantham [15] by establishing the existence of coupled coincidence points, using a pair of commutative maps. This proved to be a foundation stone in the development of fixed point theory with applications to partially ordered sets. Since then much work has been done in this direction by different authors. For more details the reader may consult ([17-28]). Recently, the problems concerning the computation of coupled fixed points in metric space has been fuzzified, see [29–34].

In this paper, we establish coupled coincidence point and coupled fixed point results for a pair of mappings under the assumption of a new inequality in the setting of fuzzy metric space having Hadžić type t-norm. As an application, a corresponding result in the metric space is also obtained. As a matter of fact, in our results we do not require the pair of the mappings to be commuting or compatible nor we have assumed the self mapping $g: X \rightarrow X$ to be monotonic, further, the completeness of the space X has also been replaced by the completeness of the range subspaces of any one of the mappings F or g, hence our results generalize a number of existing results present in the literature in general. Also, various existing results on coupled coincidence points and coupled fixed points are extended, for details one can refer to [23, 38, 39].

2. Preliminaries

Definition 2.1 [6]. "The 3-tuple (Ӽ, Ɱ, ∗) is called a GV-FMS, if Ӽ is an arbitrary set, ∗ is a continuous t-norm and Ɱ is a fuzzy set on $Ӽ^2 \times (0,\infty)$ satisfying the following conditions:

(FⱮ-1) Ɱ(ӿ, ӯ, 0) > 0,
(FⱮ-2) Ɱ(ӿ, ӯ, ț) = 1 iff ӿ = ӯ,
(FⱮ-3) Ɱ(ӿ, ӯ, ț) = Ɱ(ӯ, ӿ, ț),
(FⱮ-4) Ɱ(ӿ, ӯ, ț) ∗ Ɱ(ӯ, z, s) ≤ Ɱ(ӿ, z, ț + s),
(FⱮ-5) Ɱ(ӿ, ӯ, .) : (0, ∞) → [0, 1] is continuous for all ӿ, ӯ, z ∈ Ӽ and s, ț > 0".

Remark 2.2. (i) In present work, we consider (Ӽ, Ɱ, ∗) be a GV-FMS with a Hadžić type t-norm, Ɱ(ӿ, ӯ, ț) → 1 as ț → ∞ for all ӿ, ӯ ∈ Ӽ and ≼ be a partial order defined on Ӽ, then, we write it as (Ӽ, Ɱ, ∗, ≼) and in short, we call it as PO-GV-FMS.

(ii) Unless otherwise stated we consider the mappings Ᵽ: Ӽ × Ӽ → Ӽ, g: Ӽ → Ӽ.

Definition 2.3[6]. Let (Ӽ, Ɱ, ∗) be a GV-FMS, then

(i) "a sequence $\{ӿ_n\}$ in Ӽ is said to be Convergent to a point ӿ ∈ Ӽ, if $\lim_{n\to\infty} M(ӿ_n, ӿ, ț) = 1$, for all ț > 0";

(ii) "a sequence $\{ӿ_n\}$ in Ӽ is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and ț > 0, there exists a positive integer n_0 such that $M(ӿ_n, ӿ_m, ț) > 1 - \varepsilon$ for each n, m ≥ n_0";

(iii) "a fuzzy metric space in which every Cauchy sequence is convergent is said to be complete".

Definition 2.4 ([16]). "Let (Ҳ, ≼) be a partially ordered set (POS) and Ꝑ, ɡ be the mappings. We say Ꝑ has the mixed ɡ-monotone property (shortly, MɡMP) if Ꝑ(ҳ, ɏ) is monotone ɡ-non-decreasing in its first argument and is monotone ɡ-non-increasing in its second argument".

Taking ɡ to be identity mapping in above definition, we can obtain the definition of mixed monotone property (shortly, MMP).

Definition 2.5 ([16]). "An element (ҳ, ɏ) ∈ Ҳ × Ҳ, is called a coupled coincidence point (CCP) of the mappings Ꝑ and ɡ if Ꝑ(ҳ, ɏ) = ɡҳ and Ꝑ(ɏ, ҳ) = ɡɏ".

Taking ɡ to be the identity mapping in the above mapping, we can obtain the definition of coupled fixed point (CFP).

Definition 2.6 ([16]). "An element (ҳ, ɏ) ∈ Ҳ × Ҳ, is called a coupled common fixed point (CCFP) of the mappings Ꝑ and ɡ if ҳ = ɡҳ = Ꝑ(ҳ, ɏ) and ɏ = ɡɏ = Ꝑ(ɏ, ҳ)".

Lemma 2.7 [31]. "Let (Ҳ, Ɱ, ∗) be a fuzzy metric space with a Hadžić type t-norm ∗ such that Ɱ(ҳ, ɏ, ţ) → 1 as ţ → ∞ for all ҳ, ɏ ∈ Ҳ. If the sequences $\{ҳ_n\}$ and $\{ɏ_n\}$ in Ҳ are such that, for all n ≥ 1, t > 0, M($ҳ_n$, $ҳ_{n+1}$, ţ) ∗ M($ɏ_n$, $ɏ_{n+1}$, ţ) ≥ M($ҳ_{n-1}$, $ҳ_n$, ţ$/k$) ∗ M($ɏ_{n-1}$, $ɏ_n$, ţ$/k$) where 0 < k < 1, then the sequences $\{ҳ_n\}$ and $\{ɏ_n\}$ are Cauchy sequences".

Definition 2.8. Let (Ҳ, ḓ) be a metric space endowed with a partial ordering (≼). We say Ҳ is regular if it satisfies the following property:

(i) "if a non-decreasing sequence $\{ҳ_n\}$ → ҳ, then $ҳ_n$ ≼ ҳ for all $n \geq 0$"; (1.1)

(ii) "if a non-decreasing sequence $\{ɏ_n\}$ → ɏ, then ɏ ≼ $ɏ_n$ for all $n \geq 0$". (1.2)

Remark 2.9. Note that if (Ҳ, Ɱ, ∗) be a GV-FMS endowed with a partial ordering (≼). We say that Ҳ is regular if it satisfies properties (1.1) and (1.2) with respect to the convergence in a GV-FMS.

3. Main Results

This section focuses on main coupled coincidence point results in fuzzy metric spaces.

Theorem 3.1. Let (Ҳ, Ɱ, ∗, ≼) be a PO-GV-FMS. Let (Ꝑ, ꞡ) be pair of mappings such that Ꝑ has MgMP and satisfies the following conditions:

(Ҳ - 1) Ꝑ(Ҳ × Ҳ) ⊆ ꞡ(Ҳ);
(Ҳ - 2) one of the range subspaces Ꝑ(Ҳ × Ҳ) or ꞡ(Ҳ) is complete;
(Ҳ - 3) there exists ꝁ in (0, 1) such that

$$Ɱ(Ꝑ(ꭗ, ꭚ), Ꝑ(µ, ꝟ), ꝁţ) ∗ Ɱ(Ꝑ(ꭚ, ꭗ), Ꝑ(ꝟ, µ), ꝁţ) \geq Ɱ(ꞡ(ꭗ), ꞡ(µ), ţ) ∗ Ɱ(ꞡ(ꭚ), ꞡ(ꝟ), ţ),$$

for all ꭗ, ꭚ, µ, ꝟ in Ҳ and ţ > 0 with ꞡ(ꭗ) ≼ ꞡ(µ) and ꞡ(ꭚ) ≽ ꞡ(ꝟ) (or, ꞡ(ꭗ) ≽ ꞡ(µ) and ꞡ(ꭚ) ≼ ꞡ(ꝟ));

(Ҳ - 4) there exist two elements $ꭗ_0, ꭚ_0$ in Ҳ such that the following satisfies:

$ꞡ(ꭗ_0) ≼ Ꝑ(ꭗ_0, ꭚ_0)$ and $ꞡ(ꭚ_0) ≽ Ꝑ(ꭚ_0, ꭗ_0)$ (or, $ꞡ(ꭗ_0) ≽ Ꝑ(ꭗ_0, ꭚ_0)$ and $ꞡ(ꭚ_0) ≼ Ꝑ(ꭚ_0, ꭗ_0)$);

Further, suppose either

(a) both the mappings Ꝑ and ꞡ are continuous, or

(b) Ҳ is regular;

then, the (Ꝑ, ꞡ) has a CCP in Ҳ.

Proof. Without loss of generality, by (Ҳ - 4) suppose that $ꭗ_0, ꭚ_0$ in Ҳ be such that $ꞡ(ꭗ_0) ≼ Ꝑ(ꭗ_0, ꭚ_0)$ and $ꞡ(ꭚ_0) ≽ Ꝑ(ꭚ_0, ꭗ_0)$. Since Ꝑ(Ҳ × Ҳ) ⊆ ꞡ(Ҳ) and Ꝑ has ⱮꞡⱮP so inductively, the sequences $\{ꭗ_n\}$ and $\{ꭚ_n\}$ can be constructed in Ҳ such that

ꞡ(ӽ$_{n+1}$) = Ꝑ(ӽ$_n$, ȝ$_n$) and ꞡ(ȝ$_{n+1}$) = Ꝑ(ȝ$_n$, ӽ$_n$) for all $n \geq 0$. (3.1)
Further, by using (3.1) and since Ꝑ has ⱮꞡⱮP, inductively, it can be shown that

ꞡ(ӽ$_n$) ≼ ꞡ(ӽ$_{n+1}$) and ꞡ(ȝ$_n$) ≽ ꞡ(ȝ$_{n+1}$) for all $n \geq 0$. (3.2)

We suppose either ꞡ(ӽ$_{n+1}$) = Ꝑ(ӽ$_n$, ȝ$_n$) ≠ ꞡ(ӽ$_n$) or ꞡ(ȝ$_{n+1}$) = Ꝑ(ȝ$_n$, ӽ$_n$) ≠ ꞡ(ȝ$_n$) for all $n \geq 0$, otherwise, the we can obtain directly the CCP of the pair (Ꝑ, ꞡ).

Using (3.1) and (3.2), from (Χ - 3), for ț > 0, n > 0, we have

Ɱ(ꞡ(ӽ$_n$), ꞡ(ӽ$_{n+1}$), ꝁț) ∗ Ɱ(ꞡ(ȝ$_n$), ꞡ(ȝ$_{n+1}$), ꝁț)

= Ɱ(Ꝑ(ӽ$_{n-1}$, ȝ$_{n-1}$), Ꝑ(ӽ$_n$, ȝ$_n$), ꝁț) ∗ Ɱ(Ꝑ(ȝ$_{n-1}$, ӽ$_{n-1}$), Ꝑ(ȝ$_n$, ӽ$_n$), ꝁț)

≥ Ɱ(ꞡ(ӽ$_{n-1}$), ꞡ(ӽ$_n$), ț) ∗ Ɱ(ꞡ(ȝ$_{n-1}$), ꞡ(ȝ$_n$), ț)

≥ Ɱ(ꞡ(ӽ$_{n-1}$), ꞡ(ӽ$_n$), ț) ∗ Ɱ(ꞡ(ȝ$_{n-1}$), ꞡ(ȝ$_n$), ț). (3.3)

Using (3.3), by applying Lemma 7, the sequences {ꞡ(ӽ$_n$)} and {ꞡ(ȝ$_n$)} behaves like Cauchy sequences. Without loss of generality, suppose ꞡ(Χ) is complete, so there exist ӽ, ȝ in Χ such that for all ț > 0, we have that

$\lim_{n \to \infty}$ Ɱ(ꞡ(ӽ), ꞡ(ӽ$_n$), ț) = $\lim_{n \to \infty}$ Ɱ(ꞡ(ӽ), Ꝑ(ӽ$_n$, ȝ$_n$), ț) = 1,

$\lim_{n \to \infty}$ Ɱ(ꞡ(ȝ), ꞡ(ȝ$_n$), ț) = $\lim_{n \to \infty}$ Ɱ(ꞡ(ȝ), Ꝑ(ȝ$_n$, ӽ$_n$), ț) = 1. (3.4)

Suppose, condition (a) holds.

Define a multifunction Ǵ: ꞡ(Χ) → $2^{Χ}$ by Ǵ(ȝ) = {ӽ ∈ Χ: ꞡ(ӽ) = ȝ}. Making use of Axiom of Choice, a function h: ꞡ(Χ) → Χ, so that h(ȝ) belongs to Ǵ(ȝ) for all ȝ in ꞡ(Χ) can be constructed, yielding ꞡ(h(ȝ)) = ȝ for all ȝ in ꞡ(Χ). Consider È = {h(ȝ): ȝ ∈ ꞡ(Χ)} ⊆ Χ. Clearly, the map ꞡ: È → Χ is injective with ꞡ(È) = Ǵ(Χ).

Define another map Ƕ: ꞡ(È) × ꞡ(È) → Χ by

Ƕ(ꞡ(e), ꞡ(f)) = Ꝑ(e, f) for ꞡ(e), ꞡ(f) in ꞡ(È) (=ꞡ(Χ)). (3.5)

Since ꞡ: È → Χ is injective, the map Ƕ is well-defined. By (3.4) and (3.5),

$$\lim_{n\to\infty} H(g(x_n), g(y_n)) = \lim_{n\to\infty} P(x_n, y_n) = \lim_{n\to\infty} g(x_n) = g(x),$$
$$\lim_{n\to\infty} H(g(y_n), g(x_n)) = \lim_{n\to\infty} P(y_n, x_n) = \lim_{n\to\infty} g(y_n) = g(y). \qquad (3.6)$$
Using the continuity of P and g, the continuity of the map H can be achieved, so by (3.6) we have
$$H(g(x), g(y)) = g(x) \quad \text{and} \quad H(g(y), g(x)) = g(y). \qquad (3.7)$$
On combining (3.5) with (3.7), we see that $P(x, y) = g(x)$ and $P(y, x) = g(y)$.

Next, we show the result for the condition (b).

By (3.2) and (3.4), we have
$$g(x_n) \preccurlyeq g(x) \quad \text{and} \quad g(y) \preccurlyeq g(y_n) \quad \text{for } n \geq 0. \qquad (3.8)$$
Suppose that $(g(x_n), g(y_n)) \neq (g(x), g(y))$ for $n \geq 0$ otherwise (x_n, y_n) is a CCP of the pair (P, g). Now, for $t > 0$, $n \geq 0$, we obtain
$$M(P(x, y), g(x), t) \geq M(P(x, y), P(x_n, y_n), kt) * M(P(x_n, y_n), g(x), t - kt), \qquad (3.9)$$
$$M(P(y, x), g(y), t) \geq M(P(y, x), P(y_n, x_n), kt) * M(P(y_n, x_n), g(y), t - kt). \qquad (3.10)$$
From (3.9) and (3.10), for all $t > 0$, we obtain that
$$\begin{aligned}
&M(P(x, y), g(x), t) * M(P(y, x), g(y), t)\\
&\geq [M(P(x, y), P(x_n, y_n), kt) * M(P(x_n, y_n), g(x), t - kt)]\\
&* [M(P(y, x), P(y_n, x_n), kt) * M(P(y_n, x_n), g(y), t - kt)]\\
&= [M(P(x, y), P(x_n, y_n), kt) * M(P(y, x), P(y_n, x_n), kt)]\\
&* [M(P(x_n, y_n), g(x), t - kt) * M(P(y_n, x_n), g(y), t - kt)]. \qquad (3.11)
\end{aligned}$$
Since $g(x_n) \preccurlyeq g(x)$ and $g(y) \preccurlyeq g(y_n)$ for all $n \geq 0$, using $(X - 3)$, we obtain that
$$\begin{aligned}
&M(P(x, y), P(x_n, y_n), kt) * M(P(y, x), P(y_n, x_n), kt)]\\
&\geq M(g(x), g(x_n), t) * M(F(g(y), g(y_n), t). \qquad (3.12)
\end{aligned}$$
By (3.11) and (3.12), we obtain that
$$M(P(x, y), g(x), t) * M(P(y, x), g(y), t)$$

$\geq (M(g(x), g(x_n), t) * M(F(g(y), g(y_n), t))$

$* [M(P(x_n, y_n), g(x), t - kt) * M(P(y_n, x_n), g(y), t - kt)]$. (3.13)

Letting limit as $n \to \infty$ in (3.13), for $t > 0$, we obtain that

$M(P(x, y), g(x), t) * M(P(y, x), g(y), t)$

$\geq \lim_{n \to \infty} [M(g(x), g(x_n), t) * M(F(g(y), g(y_n), t)]$

$* \lim_{n \to \infty} [M(P(x_n, y_n), g(x), t - kt) * M(P(y_n, x_n), g(y), t - kt)]$.

By (3.4),

$$M(P(x, y), g(x), t) * M(P(y, x), g(y), t)$$
$$\geq (1 * 1) * (1 * 1) = 1.$$

That is, $M(P(x, y), g(x), t) * M(P(y, x), g(y), t) \geq 1$.

Therefore, $M(P(x, y), g(x), t) = 1$ and $M(P(y, x), g(y), t) = 1$, which implies $g(x) = P(x, y)$ and $g(y) = P(y, x)$.

Hence, we obtained the result.

Next, we give some examples in support of Theorem 3.1.

Example 3.2. Let $(X, \preccurlyeq)$ is the POS with $X = [0, 1)$ with $\leq$ being PO $\preccurlyeq$, the natural ordering of real numbers. Define $M(x, y, t) = \frac{t}{t + |x - y|}$ for x, y in X and $t > 0$ and $x * y = \min\{x, y\}$ for all x, y in $[0, 1]$. Then $(X, M, *, \preccurlyeq)$ be a PO-GV-FMS, which is not complete. Also, X is regular. Define the mappings $P: X \times X \to X$ and $g: X \to X$ respectively by

$$P(x, y) = 0.4 \text{ for all } (x, y) \text{ in } X \times X$$

and

$$g(x) = \begin{cases} 0.7 & if \ \ 0 \leq x < 0.5, \\ x - 0.3 & if \ \ 0.5 \leq x < 1. \end{cases}$$

Since $g(P(x, y)) = g(0.4) = 0.7 \neq 0.4 = P(g(x), g(y))$ for all x, y in X, the mappings P and g are neither commutative nor compatible.

The following observations are immediate:

(i) $P(X \times X) \subseteq g(X)$;

(ii) $g(X)$ is complete;
(iii) the map g is not continuous;
(iv) P has $MgMP$;
(v) the map g is not monotonic.

Further, there exist $x_0 = 0.3$ and $y_0 = 0.5$ such that $g(x_0) = g(0.3) = 0.7 \geq 0.4 = P(0.3, 0.5) = P(x_0, y_0)$ and $g(y_0) = g(0.5) = 0.2 \leq 0.4 = P(0.5, 0.3) = P(y_0, x_0)$.

Also, the inequality $(X - 3)$ holds, since for any k in $(0, 1)$, $t > 0$

$M(P(x, y), P(u, v), kt) * M(P(y, x), P(v, u), kt) = 1 * 1$
$\geq M(g(x), g(u), t) * M(g(y), g(v), t)$,
for all x, y, u, v in X, with $g(x) \leq g(u)$ and $g(y) \geq g(v)$.

Hence, all hypotheses of Theorem 3.1 hold, thus, the pair (P, g) has a CCP in $X \times X$. Thus on applying Theorem 3.1, we get that the point (0.7, 0.7) is a CCP of the pair (P, g).

Corollary 3.3. Let $(X, M, *, \preccurlyeq)$ be a PO-GV-FMS which is complete. Let the mapping P has MMP and subjected to following:

$(X - 5)$ there exists k in $(0, 1)$ such that

$M(P(x, y), P(\mu, v), kt) * M(P(y, x), P(v, \mu), kt)$

$\geq M(x, \mu, t) * M(y, v, t)$,
for all x, y, μ, v in X, $t > 0$ with $x \preccurlyeq \mu$ and $y \succcurlyeq v$ (or, $x \succcurlyeq \mu$ and $y \preccurlyeq v$).

Further, suppose either

(a) the mapping P is continuous, or
(b) X is regular.

If there exist x_0, y_0 in X such that $x_0 \preccurlyeq P(x_0, y_0)$ and $y_0 \succcurlyeq P(y_0, x_0)$ (or, $x_0 \succcurlyeq P(x_0, y_0)$ and $y_0 \preccurlyeq P(y_0, x_0)$), then the map P has a CFP in X.

Proof. Proof follows by considering $g = I_X$ (identity map on X), in Theorem 3.1.

4. Existence and Uniqueness of CCFP

Very recently, in order to obtain CCFP for nonlinear contractive mappings in cone metric spaces, Abbas et al. [41] gave the notion of w-compatible mappings. In this section, utilizing the notion of w-compatible mappings we extend Theorem 3.1 to ensure the existence and uniqueness of the CCFP in fuzzy metric space.

Definition 4.1 ([41]). The mappings Ꝑ and g are called w-compatible if g(Ꝑ(x, y)) = Ꝑ(gx, gy) whenever g(x) = Ꝑ(x, y) and g(y) = Ꝑ(y, x) for x, y ∈ X.

Now, we give our result as follows:

Theorem 4.2. If in Theorem 3.1 an additional hypothesis that: "for every (x, y), (x', y') in X × X, there exists a (μ, v) in X × X such that (Ꝑ(μ, v), Ꝑ(v, u)) is comparable to (Ꝑ(x, y), Ꝑ(y, x)) and $(Ꝑ(x', y'), Ꝑ(y', x')$. Let the pair (Ꝑ, g) is w-compatible, then it has a unique CCFP.

Proof. Using Theorem 3.1, set of coupled coincidences of pair (Ꝑ, g) is not empty. For obtaining the result, we first show: if (x, y) and (x', y') are CCP, then

$g(x) = g(x')$ and $g(y) = g(y')$. (4.1)

By assumption there is (μ, v) in X × X such that (Ꝑ(μ, v), Ꝑ(v, v)) is comparable with (Ꝑ(x, y), Ꝑ(y, x)) and $(Ꝑ(x', y'), Ꝑ(y', x'))$. Put $\mu_0 = \mu$, $v_0 = v$ and choose μ_1, v_1 in X so that $g(\mu_1) = Ꝑ(\mu_0, v_0)$, $g(v_1) = Ꝑ(v_0, \mu_0)$.

Then, like in Theorem 3.1, sequences $\{g(\mu_n)\}$ and $\{g(v_n)\}$ can be defined such that $g(\mu_{n+1}) = Ꝑ(\mu_n, v_n)$ and $g(v_{n+1}) = Ꝑ(v_n, \mu_n)$.

Further, set $x_0 = x$, $y_0 = y$, $x'_0 = x'$, $y'_0 = y'$, we construct sequences $\{g(x_n)\}$, $\{g(y_n)\}$ and $\{g(x'_n)\}$, $\{g(y'_n)\}$ following the same track. Then, it can be easily obtained that

$g(x_{n+1}) = Ꝑ(x_n, y_n)$, $g(y_{n+1}) = Ꝑ(y_n, x_n)$

and

$g(x'_{n+1}) = P(x'_n, y'_n)$, $g(y'_{n+1}) = P(y'_n, x'_n)$, for $n \geq 0$.

Since $(P(\mu, v), P(v, \mu)) = (g(\mu_1), g(v_1))$ and $(g(x), g(y)) = (P(x, y), P(y, x)) = (g(x_1), g(y_1))$ are comparable, then $g(\mu_1) \succcurlyeq g(x)$ and $g(v_1) \preccurlyeq g(y)$. Easily, it can be shown $(g(\mu_n), g(v_n))$ and $(g(x), g(y))$ are comparable; that is, $g(\mu_n) \succcurlyeq g(x)$ and $g(v_n) \preccurlyeq g(y)$ for all $n > 0$. Thus, from $(X - 3)$, for $n > 0$,

$M(g(\mu_{n+1}), g(x), kt) * M(g(v_{n+1}), g(y), kt)$

$= M(P(\mu_n, v_n), P(x, y), kt) * M(P(v_n, \mu_n), P(y, x), kt)$

$\geq M(g(\mu_n), g(x), t) * M(g(v_n), g(y), t)$,

that is, for all $n \in \mathbb{N}$, we have that

$M(g(\mu_{n+1}), g(x), kt) * M(g(v_{n+1}), g(y), kt)$

$\geq M(g(\mu_n), g(x), t) * M(g(v_n), g(y), t)$. (4.2)

We claim that

$\lim_{n\to\infty} g(\mu_n) = g(x)$ and $\lim_{n\to\infty} g(v_n) = g(y)$. (4.3)

If $\lim_{n\to\infty} g(\mu_n) = \mu$ and $\lim_{n\to\infty} g(v_n) = v$ for some μ, v in X, it suffices to obtain that $\mu = g(x)$ and $v = g(y)$. Letting limit as $n \to \infty$ in (4.2) and using the properties of γ, we obtain that

$M(\mu, g(x), kt) * M(v, g(y), kt) \geq M(\mu, g(x), t) * M(v, g(y), t)$

$\geq M(\mu, g(x), t) * M(v, g(y), t)$. (4.4)

Thus,

$M(\mu, g(x), t) * M(v, g(y), t) \geq M(\mu, g(x), t/k^n) * M(v, g(y), t/k^n)$,

for all $n \in \mathbb{N}$, implying that $M(\mu, g(x), t) * M(v, g(y), t) = 1$ for $t > 0$. It follows $M(\mu, g(x), t) = M(v, g(y), t) = 1$ for all $t > 0$, therefore we can obtain that $\mu = g(x)$ and $v = g(y)$, and hence (4.3) holds.

Similarly, we obtain that

$\lim_{n\to\infty} g(\mu_n) = g(x')$ and $\lim_{n\to\infty} g(v_n) = g(y)$. (4.5)

Using uniqueness of limit, it is obtainable that $g(x) = g(x')$ and $g(y) = g(y')$. Thus, we have proved (4.1).

As $g(x) = P(x, y)$, $g(y) = P(y, x)$ and the pair (P, g) is w-compatible, it is obtainable that

$g(g(x)) = g(P(x, y)) = P(g(x), g(y))$ and $g(g(y)) = g(P(y, x)) = P(g(y), g(x))$. (4.6)

Denote $g(x) = z$, $g(y) = w$. Then, using (4.6),

$g(z) = P(z, w)$ and $g(z) = P(w, z)$. (4.7)

Thus, (z, w) is a CCP.

Then, by (4.1) with $x' = z$ and $y' = w$, we get $g(z) = g(x)$ and $g(w) = g(y)$;

that is, $g(z) = z$, $g(w) = w$. (4.8)

Using (4.7) and (4.8), we obtain

$$z = g(z) = P(z, w) \text{ and } w = g(w) = P(w, z).$$

Hence, (z, w) is CCFP of P and g.

For uniqueness, let (p, q) be any CCFP. Then, by (4.1), we get

$p = g(p) = g(z) = z$ and $q = g(q) = g(w) = w$.

5. Application in metric space

In this section, as application of the results proved in the earlier sections of this paper, we obtain CCP results in the framework of ordered metric spaces.

Theorem 5.1. Let $(X, \preccurlyeq)$ be POS and d a metric on X so that (X, d) is a metric space, the pair (P, g) of maps be such that P has MgMP and subjected to following:

(X - 6) there exists some k in (0, 1) such that

$$\max\{d(P(x, y), P(\mu, v)), d(P(y, x), P(v, \mu))\} \leq \frac{k}{2}[d(g(x), g(\mu)) + d(g(y), g(v))],$$

for all x, y, μ, v in X for which $g(x) \preccurlyeq g(\mu)$, $g(y) \succcurlyeq g(v)$ (or, $g(x) \succcurlyeq g(\mu)$ and $g(y) \preccurlyeq g(v)$).

Suppose that (X - 1), (X - 2) and (X - 4) holds. Also, assume that

(a) both the maps P and g are continuous, or

(b) X is regular,

then the pair (P, g) has a CCP.

Proof. For all x, y in X and $t > 0$, define $M(x, y, t) = \frac{t}{t+d(x,\ y)}$ and $p * q = \min\{p, q\}$ with p, q in [0, 1]. Then, (X, M, $*$) is FMS, $*$ as the Hadžić type t-norm. Also, it can be easily seen that $M(x, y, t) = \frac{t}{t+d(x,\ y)} \rightarrow 1$ as $t \rightarrow \infty$, for all x, y in X. We next verify that the inequality (X - 6) implies (X - 3). If otherwise, for $\gamma(t) = t$, from (X - 3), for some $t > 0$ and x, y, μ, v in X with $g(x) \preccurlyeq g(\mu)$, $g(y) \succcurlyeq g(v)$, we have

$$\min\{(t/(t + \tfrac{1}{k}\, d(P(x, y), P(\mu, v)))), (t/(t + \tfrac{1}{k}\, d(P(y, x), P(v, \mu))))\}$$
$$< \min\{(t/(t + d(g(x), g(\mu)))), (t/(t + d(g(y), g(v))))\},$$

then, we have either,

$$(t/(t + \tfrac{1}{k}\, d(P(x,\ y),\ P(\mu,\ v)))) < \min\{(t/(t + d(g(x),\ g(\mu)))),\ (t/(t + d(g(y), g(v))))\}, \qquad (5.1)$$

or,

$$(t/(t + \tfrac{1}{k}\, d(P(y,\ x),\ P(v,\ \mu)))) < \min\{(t/(t + d(g(x),\ g(\mu)))),\ (t/(t + d(g(y), g(v))))\}. \qquad (5.2)$$

From (5.1), it follows

$$t + \tfrac{1}{k}\, d(P(x, y), P(\mu, v)) > t + d(g(x), g(\mu)). \qquad (5.3)$$

$$t + \tfrac{1}{k}\, d(P(x, y), P(\mu, v)) > t + d(g(y), g(v)). \qquad (5.4)$$

Combining (5.3) and (5.4), we attain

$$d(P(x, y), P(\mu, v)) > \tfrac{k}{2}\, [d(g(x), g(\mu)) + d(g(y), g(v))]. \qquad (5.5)$$

Similarly, from (5.2), we obtain that

$$d(P(y, x), P(v, \mu)) > \tfrac{k}{2}\, [d(g(x), g(\mu)) + d(g(y), g(v))]. \qquad (5.6)$$

Using (5.5) and (5.6), we have that

max{ꝱ(Ꝑ(ꭓ, ꝩ), Ꝑ(μ, ꝟ)), ꝱ(Ꝑ(ꝩ, ꭓ), Ꝑ(ꝟ, μ))} > $\frac{ꝁ}{2}$ [[ꝱ(ꞡ(ꭓ), ꞡ(μ)) + ꝱ(ꞡ(ꝩ), ꞡ(ꝟ))]],
which is a contradiction to (Ꭓ - 6). Now, on applying Theorem 3.1, we get desired result.

Theorem 5.2. Let (Ꭓ, ≼) be POS and ꝱ a metric on Ꭓ so that (Ꭓ, ꝱ) is a metric space. Let the pair (Ꞙ, ꞡ) of maps be such that Ꝑ has ꟽꞡꟽP and subjected to following:
(Ꭓ - 7) there exists some ꝁ in (0, 1) such that
ꝱ(Ꝑ(ꭓ, ꝩ), Ꝑ(μ, ꝟ)) + ꝱ(Ꝑ(ꝩ, ꭓ), Ꝑ(ꝟ, μ)) ≤ ꝁ [ꝱ(ꞡ(ꭓ), ꞡ(μ)) + ꝱ(ꞡ(ꝩ), ꞡ(ꝟ))],

for all ꭓ, ꝩ, μ, ꝟ in Ꭓ for which ꞡ(ꭓ) ≼ ꞡ(μ), ꞡ(ꝩ) ≽ ꞡ(ꝟ) (or, ꞡ(ꭓ) ≽ ꞡ(μ) and ꞡ(ꝩ) ≼ ꞡ(ꝟ)).
Suppose that (Ꭓ - 1), (Ꭓ - 2) and (Ꭓ - 4) holds. Also, assume that

(a) both the maps Ꝑ and ꞡ are continuous, or
(b) Ꭓ is regular,

then, the pair (Ꝑ, ꞡ) has a CCP.
Proof. Result follows immediately by considering the well known fact that, $\frac{ꭓ+ꝩ}{2}$ ≤ max{ꭓ,ꝩ} in Theorem 5.1.

Remark 5.3. (i) Theorem 5.1 provides an extension and generalization of a result of Berinde [Corollary 1, 38] for a pair of mappings.

(ii) Theorem 5.2 improves the recent result of Jain et al. [Corollary 2.3, 23]. By taking ꞡ = $I_{Ꭓ}$ (the identity mapping on Ꭓ) in Theorem 5.2, we obtain the result of Berinde [Theorem 3, 39].

Remark 5.4. Unique coupled common fixed point for the pair (Ꝑ, ꞡ) under the hypotheses of Theorem 5.1 (or Theorem 5.2) can be obtained by assuming the additional assumptions as in Theorem 4.2.

References

[1] Zadeh LA: Fuzzy Sets, Information and Control, 89 (1965), 338-353.

[2] Deng ZK: Fuzzy pseudo metric spaces, J. Math. Anal. Appl. 86(1982), 74-95.

[3] Erceg MA: Metric spaces in fuzzy set theory, J. Math. Anal. Appl. 69 (1979), 205-230.

[4] Kaleva O, Seikkala S: On fuzzy metric spaces, Fuzzy Sets Syst. 12 (1984), 215-229.

[5] Kramosil I, Michalek J: Fuzzy Metric and Statistical metric Spaces, Kybernetica, 11 (1975), 326-334.

[6] George A, Veeramani P: On some results in fuzzy metric spaces, Fuzzy Sets Syst. 64 (3) (1994), 395-399.

[7] Grabiec M: Fixed points in fuzzy metric spaces, Fuzzy Sets Syst. 27 (3) (1988), 385-389.

[8] Fang JX: On fixed point theorems in fuzzy metric spaces, Fuzzy Sets Syst. 46(1992), 107-113.

[9] Edelstein M: On fixed and periodic points under contraction mappings, J. Lond. Math. Soc. 37 (1962), 74-79.

[10] Istratescu I: A fixed point theorem for mappings with a probabilistic contractive iterate, Rev. Roum. Math. Pures Appl. 26(1981), 431-435.

[11] Sehgal VM, Bharucha-Reid AT: Fixed points of contraction mappings on PM-spaces, Math. Syst. Theory 6(1972), 97-100.

[12] George A, Veeramani P: On some results of analysis for fuzzy metric spaces, Fuzzy Sets Syst. 90(1997), 365-368.

[13] Singh B, Chauhan MS: Common fixed points of compatible maps in fuzzy metric spaces, Fuzzy Sets Syst. 115(2000), 471-475.

[14] Gregori V, Sapena A: On fixed-point theorems in fuzzy metric spaces, Fuzzy Sets Syst. 125(2002), 245-252 (2002).

[15] Bhaskar TG, Lakshmikantham V: Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. 65(7) (2006), 1379–1393.

[16] Lakshmikantham V, Ćirić Lj. B: Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal. 70(2009), 4341–4349.

[17] Choudhury BS, Kundu A: A coupled coincidence point result in partially ordered metric spaces for compatible mappings, Nonlinear Anal. 73(2010), 2524–2531.

[18] Luong NV, Thuan NX: Coupled fixed point in partially ordered metric spaces and applications, Nonlinear Anal. 74(2011), 983–992.

[19] Samet B, Vetro C: Coupled fixed point theorems for multi-valued nonlinear contraction mappings in partially ordered metric spaces, Nonlinear Anal. 74 (12) (2011), 4260-4268.

[20] Shatanawi W, Samet B, Abbas M: Coupled fixed point theorems for mixed monotone mappings in ordered partial metric spaces, Math. Comput. Modelling, doi:10.1016/j.mcm.2011.08.042.

[21] Aydi H, Karapınar E, Shatanawi W: Coupled fixed point results for (ψ, ϕ) - weakly contractive condition in ordered partial metric spaces, Comput. Math. Appl. 62(12) (2011), 4449–4460.

[22] Gordji ME, Cho YJ, Ghods S, Ghods M, Dehkordi MH: Coupled fixed point theorems for contractions in partially ordered metric spaces and applications, Math. Probl. Eng. Vol. 2012, Article ID 150363, 20 pages, 2012.

[23] Jain M, Tas K, Kumar S, Gupta N: Coupled common fixed points involving a (φ, ψ) - contractive condition for mixed g-monotone operators in partially ordered metric spaces, J. Inequal. Appl. 2012, **2012**:285.

[24] Jain M, Tas K, Rhoades BE, Gupta N: Coupled Fixed Point Theorems for Generalized Symmetric Contractions in Partially Ordered Metric Spaces and applications, J. Comput. Anal. Appl. 16(3) (2014), 438 – 454.

[25] Jain M, Gupta N, Vetro C, Kumar S: Coupled Fixed Point Theorems for Symmetric $(\boldsymbol{\phi}, \boldsymbol{\psi})$-weakly Contractive Mappings in Ordered Partial Metric Spaces, The Journal of Mathematics and Computer Sciences 7(4) (2013), 230 - 304.

[26] Jain M, Tas K, Kumar S, Gupta N: Coupled Fixed Point Theorems for a Pair of Weakly Compatible Maps along with *CLRg* Property in Fuzzy Metric Spaces, J. Appl. Math., vol. 2012, Article ID 961210, 13 pages, 2012. doi:10.1155/2012/961210.

[27] Jain M, Tas K, Gupta N: Coupled common fixed point results involving (φ,ψ)-contractions in ordered generalized metric spaces with application to integral equations, J. Inequal. Appl. 2013 **2013**:372.

[28] Jain M, Tas K: A Unique Coupled Common Fixed Point Theorem for Symmetric (φ, ψ)-Contractive Mappings in Ordered G -Metric Spaces with Applications, J. Appl. Math., vol. 2013, Article ID 134712, 13 pages, 2013. doi:10.1155/2013/134712.

[29] Hu, X-Q: Common coupled fixed point theorems for contractive mappings in fuzzy metric spaces. Fixed Point Theory Appl. 2011, Article ID 363716 (2011).

[30] Sedghi S, Altun I, Shobe N: Coupled fixed point theorems for contractions in fuzzy metric spaces, Nonlinear Anal. 72(2010), 1298-1304.

[31] Choudhury BS, Das K, Das P: Coupled coincidence point results for compatible mappings in partially ordered fuzzy metric spaces, Fuzzy Sets Syst. 222(2013), 84-97.

[32] Wang et al.: Common fixed point theorems for nonlinear contractive mappings in fuzzy metric spaces, Fixed Point Theory Appl. 2013 **2013**:191.

[33] Hu et al.: Common coupled fixed point theorems for weakly compatible mappings in fuzzy metric spaces, Fixed Point Theory Appl. 2013 **2013**:220.

[34] Jain M, Kumar S, Chugh R: Coupled fixed point theorems for weak compatible mappings in fuzzy metric spaces, Ann. Fuzzy Math. Inform. 5 (2013), No. 2, 321-336.

[35] Schweizer B, Sklar A: Probabilistic Metric Spaces, North Holland Series in Probability and Applied Math., 5 (1983).

[36] Hadžić Oand Pap E: Fixed Point Theory in Probabilistic Metric Spaces, Vol.536 of Mathematics and its Applications, Kluwer Academic, Dordrecht, The Netherlands, 2001.

[37] Rodríguez López J, Romaguera S: The Hausdorff fuzzy metric on compact sets, Fuzzy Sets Syst. 147 (2004), 273-283.

[38] Berinde V: Coupled coincidence point theorems for mixed monotone nonlinear operators, Comput. Math. Appl., 64 (2012), 1770-1777.

[39] Berinde V: Generalized coupled fixed point theorems for mixed monotone mappings partially ordered metric spaces, Nonlinear Anal. 74(2011), 7347-7355.

[40] Haghi RH, Rezapour Sh, Shahzad N, Some fixed point generalizations are not real generalizations, Nonlinear Anal. 74 (2011), 1799-1803.

[41] Abbas, M, Ali Khan, M, Radenović, S: Common coupled fixed point theorems in cone metric spaces for w-compatible mappings, Appl Math Comput. 217(2010), 195–202.

Ozimamoğlu, H. (2023). On hyper complex numbers with higher order Pell numbers components. The Journal of Analysis, 1–15. Doi:10.1007/s41478-023-00579-2

Özkan, E. & Uysal, M. (2023). On Quaternions with Higher Order Jacobsthal Numbers Components. *Gazi University Journal of Science*, *36* (1), 336-347. Doi: 10.35378/gujs. 1002454

Prasad, K., Kumari, M., Mohanta, R., & Mahato, H. (2023). The sequence of higher order Mersenne numbers and associated binomial transforms. *arXiv preprint arXiv:2307.08073*, 2023. Doi:10.48550/arXiv.2307.08073

Randić, M., Morales, D. A. & Araujo, O. (1996). Higher-order Fibonacci numbers. *Journal of Mathematical Chemistry*, 20, 79–94. Doi:10.1007/BF01165157

Shannon, A. G., Deveci, Ö. & Erdağ, Ö. (2019). Generalized Fibonacci numbers and Bernoulli polynomials. *Notes Number Theory Discrete Mathematics*, *25*(1), 193-198. Doi: 10.7546/nntdm.2019.25.1.193-198

Yılmaz, Ç. Z., & Saçlı, G. Y. (2023). On Dual Quaternions with k-Generalized Leonardo Components. *Journal of New Theory*, 44, 31-42. Doi: 10.53570/jnt.1328605

CHAPTER VIII

Mathematical Analysis of an SVEIHR Epidemic Model for the Measles Transmission Dynamics

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Introduction

Measles is an infectious respiratory disease and the disease is caused by a virus from the Paramyxoviridae family (Yanagi et al., 2006; Griffin, 2016). The incubation period for measles is usually 10-14 days. People who are exposed and infected usually recover within three weeks without any complications. However, some people exposed to the disease die or suffer from serious illness and lifelong complications (Beay, 2004; Abad and Safdar, 2015).

Annual reports on the estimated number of measles cases and deaths caused by measles worldwide are announced by the World

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Health Organization (WHO), by evaluating the reports of member countries (WHO, 2019).

Although measles is a vaccine-preventable disease, it still has alarming data globally. Measles has become a leading cause of illness and death among young children under five years of age worldwide, especially in underdeveloped countries. The spread of the virus occurs through the coughing or sneezing of an infected person. Therefore, the spread rate of the virus may be quite high. Clinical symptoms of the disease are high fever, runny nose, cough, conjunctivitis, rhinitis, small white spots, and rashes on the body of infected people.

In recent years, research on measles has become one of the most important research areas in epidemiology due to the impact of mortality rates. Many scientists are focused on finding the best ways to prevent and control the disease in the first place. Researchers who generate ideas using different models have presented their mathematical, experimental and theoretical works to the scientific world. Some of these are stated below.

Momoh et al. (2013) took into account the impact of asymptomatic individuals on measles dynamics in their study. Adewale et al. (2014) proved in their studies that the connection between people infected and uninfected with measles is effective in controlling the spread of the disease. In examining the dynamics on the effect of vaccination on measles, Smith et al. (2016) and Peter et al. (2018) conducted various research and studies.

Garba et al. (2017) created a model system that examines the dynamics of disease treatment along with the vaccination period. The effect of treatment and quarantine stages on the spread of measles was examined by Beay (2018). Zhang et al. (2010) studied the status of the epidemic versus vaccination dynamics on random graphs and scale-free networks. Mossong and Muller (2003) conducted a study on modeling the reemergence of measles within vaccinated populations.

Beay (2004) proposed a SIQR epidemic model for the disease in his study. This study also performed numerical analysis of the model to investigate the impact of treatment and quarantine on measles dynamics. Data from the study showed that quarantine and treatment combined were more effective in controlling and preventing measles. In addition, it was observed with the help of this study that the spread of measles decreased due to the treatment and quarantine of infected individuals.

The article that motivated our study is Peter et al. (2022). In this study, the literature focused on the deterministic modeling of measles disease in Nigeria. This study is based on an SVEIHR (S-susceptible, V-vaccinated, E-exposed, I-infectious, H-hospitalized, and R-recovered) model. The SVEIHR model is based on the assumption of continuous vaccination.

The findings of the current study may assist government and public health authorities in creating strategic vaccination plans to address vaccination gaps and thus prevent measles outbreaks. In our study, the numerical analysis of this type of model was examined to help prevent and control the disease. Thus, the effect of the system obtained in the stability analysis on people at different stages can be easily interpreted.

In our study, the nonstandard finite difference method (NSFD) was used as the numerical method. The method developed by Mickens (1994) can easily eliminate instabilities, unlike other numerical methods. It is very important to eliminate instabilities by choosing the denominator function (Mickens, 2002).

A lot of work has been done on the NSFD scheme. Ongun and Turhan (2012) made a numerical comparison for discrete HIV infection. For this comparison, they used the NSFD scheme in the disclosure part. Kocabıyık and Ongun (2022) also used this scheme in their distributed-order smoking model, adding one of the rare studies in this field to the literature. In their study, Kocabıyık and Ongun (2023) used the NSFD scheme in nonlinear equation systems and obtained productive results. For more detailed information, you can refer to the sources (Kocabıyık et al., 2020; Çetinkaya et al., 2021; Özdoğan and Ongun,2022).

This paper is organized as follows. In the second section, the basic information necessary for the analysis is given. This information is about the Nonstandard finite difference scheme (NSFD) scheme and stability analysis. In the third section, the system of Measles disease is defined. Again in this section, the discretization of the system was found with the NSFD method. In the fourth section, the equilibrium points of the discretized system are examined. The stability analysis of the equilibrium points was obtained in this section by finding the eigenvalues. The fifth chapter includes the results and discussion section based on the discretization and stability analysis data.

1.Basic Definitions About Nonstandard Finite Difference Scheme and Stability Analysis

In this section, information about the non-standard finite difference method and stability analysis is given.

The nonstandard finite difference method was defined by Mickens (1994). It was also applied to differential equations by Mickens. With this method, Mickens tried to solve the instabilities encountered during numerical analysis. The denominator function is used in the method to remove instabilities. With the different selection of the denominator function, instabilities are eliminated and analysis operations under different effects are made easier.

The method described by Mickens is as follows.

Definition 2.1: (Mickens, 1994) The Nonstandard finite difference scheme is defined as follows:

$$\frac{dk}{dt} \to \frac{k_{n+1} - k_n}{\phi}, t \to t_n, F(z) \to F(k_n), \quad k(t) \to k(t_n),$$

where ϕ is a denominator function. The denominator function depends on the variable p, which can be found by the equilibrium point, and the variable h, which is the step interval.

Definition 2.2: (Richter, 2002) Let the equilibrium point of the difference equation be Eq, in which case the following statements are true.

i. If the absolute values of all roots of the characteristic polynomial are less than one, the equilibrium point is locally asymptotically stable.

ii. If at least one of the absolute values of the roots of the characteristic polynomial is greater than one, the equilibrium point is unstable.

2.Discretization of Measles Transmission Dynamics

There are 6 different classifications in the population equation of the measles system. In this system, $S(t)$: the susceptible population, $V(t)$: vaccinated individuals, $E(t)$: the population exposed to the disease, $I(t)$: infected individuals, $H(t)$: the population hospitalized due to the disease, and the last $R(t)$: refers to the population that survived the disease and recovered. The ordinary form of the SVEIHR model can be given as (Peter et al., 2022):

$$\frac{dS}{dt} = \varphi - aS(t)I(t) + w\,V(t) - (\tau + \mu)S(t),$$
$$\frac{dV}{dt} = \tau S(t) - (\mu + w)V(t),$$
$$\frac{dE}{dt} = aS(t)I(t) - (\beta + \mu)E(t),$$
$$\frac{dI}{dt} = \beta E(t) - (\mu + p + \delta)I(t),$$
$$\frac{dH}{dt} = p\,I(t) - (\gamma + \delta + \mu)H(t),$$

$$\frac{dR}{dt} = \gamma H(t) - \mu R(t),$$

where, the daily intake of the susceptible population is at the rate φ. The transmission rate for the susceptible population is α, loss of immunity at the rate of vaccine decline is w and individuals in the susceptible population are vaccinated at a rate τ. Natural mortality occurs at a rate μ in all populations and the progression from the exposed class to the infected class is at the rate β. Infected people visit the hospital at a rate of ρ due to complications from measles. The measles-related mortality rate is denoted by δ and the recovery rate from the effects of measles infection following treatment is γ.

The discretization processes of the SVEIHR model are as follows:

$$\frac{S_{n+1} - S_n}{\phi_1(h)} = \varphi - aS_nI_n + wV_n - (\tau + \mu)S_{n+1},$$

$$\frac{V_{n+1} - V_n}{\phi_2(h)} = \tau S_n - (\mu + w)V_{n+1},$$

$$\frac{E_{n+1} - E_n}{\phi_3(h)} = aS_nI_n - (\beta + \mu)E_{n+1},$$

$$\frac{I_{n+1} - I_n}{\phi_4(h)} = \beta E_n - (\mu + p + \delta)I_{n+1},$$

$$\frac{H_{n+1} - H_n}{\phi_5(h)} = pI_n - (\gamma + \delta + \mu)H_{n+1},$$

$$\frac{R_{n+1} - R_n}{\phi_6(h)} = \gamma H_n - \mu R_{n+1},$$

where $\phi_i, i = 1,2, \ldots ,6$ are denominator functions and are chosen as follows:

$$\phi_1(h) = \frac{e^{(\tau+\mu)h} - 1}{(\tau + \mu)}, \phi_2(h) = \frac{e^{(\mu+w)h} - 1}{(\mu + w)}, \phi_3(h) = \frac{e^{(\beta+\mu)h} - 1}{(\beta + \mu)},$$

$$\phi_4(h) = \frac{e^{(\mu+p+\delta)h} - 1}{(\mu + p + \delta)}, \phi_5(h) = \frac{e^{(\gamma+\delta+\mu)h} - 1}{(\gamma + \delta + \mu)}, \phi_6(h) = \frac{e^{\mu h} - 1}{\mu}.$$

If the necessary arrangements are made in the following order:

$$\begin{aligned}
S_{n+1} - S_n &= \phi_1(h)(\varphi - aS_nI_n + wV_n - (\tau + \mu)S_{n+1}),\\
V_{n+1} - V_n &= \phi_2(h)(\tau S_n - (\mu + w)V_{n+1}),\\
E_{n+1} - E_n &= \phi_3(h)(aS_nI_n - (\beta + \mu)E_{n+1}),\\
I_{n+1} - I_n &= \phi_4(h)(\beta E_n - (\mu + p + \delta)I_{n+1}),\\
H_{n+1} - H_n &= \phi_5(h)(pI_n - (\gamma + \delta + \mu)H_{n+1}),\\
R_{n+1} - R_n &= \phi_6(h)(\gamma H_n - \mu R_{n+1}),
\end{aligned}$$

and then,

$$\begin{aligned}
&S_{n+1}(1 - \phi_1(h)(\tau + \mu)) = S_n - \phi_1(h)(\varphi - aS_nI_n + wV_n),\\
&V_{n+1}(1 - \phi_2(h)(\mu + w)) = V_n - \phi_2(h)\tau S_n,\\
&E_{n+1}(1 - \phi_3(h)(\beta + \mu)) = E_n - \phi_3(h)aS_nI_n,\\
&I_{n+1}(1 - \phi_4(h)(\mu + p + \delta)) = I_n - \phi_4(h)\beta E_n,\\
&H_{n+1}(1 - \phi_5(h)(\gamma + \delta + \mu)) = H_n - \phi_5(h)pI_n,\\
&R_{n+1}(1 - \phi_6(h)\mu) = R_n - \phi_6(h)\gamma H_n,
\end{aligned}$$

The final version of the discretized form of the SVEIHR system is as follows:

$$S_{n+1} = \frac{S_n - \phi_1(h)(\varphi - aS_nI_n + wV_n)}{1 - \phi_1(h)(\tau + \mu)},$$
$$V_{n+1} = \frac{V_n - \phi_2(h)\tau S_n}{1 - \phi_2(h)(\mu + w)},$$
$$E_{n+1} = \frac{E_n - \phi_3(h)aS_nI_n}{1 - \phi_3(h)(\beta + \mu)},$$
$$I_{n+1} = \frac{I_n - \phi_4(h)\beta E_n}{1 - \phi_4(h)(\mu + p + \delta)},$$
$$H_{n+1} = \frac{H_n - \phi_5(h)pI_n}{1 - \phi_5(h)(\gamma + \delta + \mu)},$$
$$R_{n+1} = \frac{R_n - \phi_6(h)\gamma H_n}{1 - \phi_6(h)\mu}.$$

If the following system of equations is solved to find the equilibrium point of the system:

$$S_n = \frac{S_n - \phi_1(h)(\varphi - aS_nI_n + wV_n)}{1 - \phi_1(h)(\tau + \mu)},$$

$$V_n = \frac{V_n - \phi_2(h)\tau S_n}{1 - \phi_2(h)(\mu + w)},$$

$$E_n = \frac{E_n - \phi_3(h)aS_nI_n}{1 - \phi_3(h)(\beta + \mu)},$$

$$I_n = \frac{I_n - \phi_4(h)\beta E_n}{1 - \phi_4(h)(\mu + p + \delta)},$$

$$H_n = \frac{H_n - \phi_5(h)pI_n}{1 - \phi_5(h)(\gamma + \delta + \mu)},$$

$$R_n = \frac{R_n - \phi_6(h)\gamma H_n}{1 - \phi_6(h)\mu}.$$

Thus, the equilibrium points are found as:

$$Eq_1 = \left(\frac{\varphi(\mu + w)}{\mu(\tau + \mu) + w(\mu + 2\tau)}, \frac{\varphi\tau}{\mu(\tau + \mu) + w(\mu + 2\tau)}, 0,0,0,0\right),$$

$$Eq_2 = \left(\frac{(\beta + \mu)\big((\mu + p + \delta)\big)}{a\beta}, \frac{\tau(\beta + \mu)\big((\mu + p + \delta)\big)}{a\beta(\mu + w)}, eq_3, eq_4, eq_5, eq_6\right).$$

Here, quite complex operations are required for the last 4 components of the Eq_2 equilibrium point. For this reason, in the stability analysis part, the components of this balance point were obtained with numerical values.

3.Stability Analysis of SVEIHR Epidemic Model

In this section, stability analysis was performed to better understand the impact of measles. Two different equilibrium points were considered for the stability analysis. The parameter values to be used in this section are expressed in the table below (Peter et al., 2022):

$\varphi: 680.27$	$\mu: 0.000309$	$\delta: 0.033720$	$\tau: 0.0000001$	$w: 0.003286$
$a: 10^{-9}$	$\beta: 0.5$	$p: 0.036246$	$\gamma: 0.062366$	$h: 0.01$

By substituting the parameters at the expressed equilibrium points, these points are obtained as follows:

$$Eq_1 = (2201459.799, 61.23671208, 0, 0, 0, 0),$$

$$Eq_2 = (70318429.96, 1956.006397, -42071.45835, -299334.4601, -112554.3$$

For stability analysis of equilibrium points, Jacobian matrix is required. The Jacobian matrix of the resulting discretized system is shown as follows:

$$\begin{pmatrix} \frac{1+\phi_1(h)(aI_n)}{1-\phi_1(h)(\tau+\mu)} & \frac{-\phi_1(h)(w)}{1-\phi_1(h)(\tau+\mu)} & 0 & \frac{\phi_1(h)(aS_n)}{1-\phi_1(h)(\tau+\mu)} & 0 & 0 \\ \frac{-\phi_2(h)\tau}{1-\phi_2(h)(\mu+w)} & \frac{1}{1-\phi_2(h)(\mu+w)} & 0 & 0 & 0 & 0 \\ \frac{-\phi_3(h)aI_n}{1-\phi_3(h)(\beta+\mu)} & 0 & \frac{1}{1-\phi_3(h)(\beta+\mu)} & \frac{-\phi_3(h)aS_n}{1-\phi_3(h)(\beta+\mu)} & 0 & 0 \\ 0 & 0 & \frac{-\phi_4(h)\beta}{1-\phi_4(h)(\mu+p+\delta)} & \frac{1}{1-\phi_4(h)(\mu+p+\delta)} & 0 & 0 \\ 0 & 0 & 0 & \frac{-\phi_5(h)p}{1-\phi_5(h)(\gamma+\delta+\mu)} & \frac{1}{1-\phi_5(h)(\gamma+\delta+\mu)} & 0 \\ 0 & 0 & 0 & 0 & \frac{-\phi_6(h)\gamma}{1-\phi_6(h)\mu} & \frac{1}{1-\phi_6(h)\mu} \end{pmatrix}$$

By substituting the parameters and the obtained equilibrium point Eq_1 into the Jacobian matrix, the characteristic equation is obtained as:

$$P(\lambda) = \lambda^6 - 5.993301\,\lambda^5 + 14.966515\,\lambda^4 - 19.933049\,\lambda^3 +$$

$$14.933067\,\lambda^2 - 5.966543\,\lambda + 0.993310$$

With the necessary calculations, the eigenvalues of this characteristic equation are in the following form:

$$\lambda_1 = 0.994983, \lambda_2 = 0.999036, \lambda_3 = 0.999322,$$

$$\lambda_4 = 0.999964, \lambda_5 = 0.999996, \lambda_6 = 0.999997.$$

Thus, with the help of Definition 2.2, the absolute values of all eigenvalues are less than 1, that is, the Eq_1 equilibrium point is locally asymptotically stable.

If the same operations are repeated for Eq_2, the characteristic equation is:

$$P(\lambda) = \lambda^6 - 5.993304\,\lambda^5 + 14.966527\,\lambda^4 - 19.933065\,\lambda^3 + 14.933077\,\lambda^2 - 5.966544\,\lambda + 0.993310$$

And the eigenvalues are,

$$\lambda_1 = 0.994307, \lambda_2 = 0.999036, \lambda_3 = 0.999956,$$

$$\lambda_4 = 0.999964, \lambda_5 = 0.999999, \lambda_6 = 1.000042.$$

As can be seen, the Eq_2 equilibrium point is unstable because the λ_6 eigenvalue does not meet the stability condition.

4.Conclusion

In this study, a stability analysis of an epidemic model of measles has been obtained. The effect of the vaccination component in the model has been examined. Discretization for numerical analysis has been obtained by the nonstandard finite difference method. Afterwards, equilibrium points were found again with this discretization method. Stability analysis of equilibrium points was examined with parameters. Thus, it has been seen that the method is suitable for this type of endemic equations. We have shown that analysis can be performed under different situations by selecting the appropriate denominator function.

References

Yanagi, Y., Takeda, M., Ohno, S. Measles virus: cellular receptors, tropism and pathogenesis. *J. Gen. Virol.* 2006;87(10):2767–2779. doi: 10.1099/vir.0.82221-0

Griffin, D. The immune response in measles: virus control, clearance and protective immunity. *Viruses.* 2016;8(10):282. doi: 10.3390/v8100282.

Beay, L. K. *AIP Conference Proceedings*. 2004 (AIP Publishing LLC, 2004).

Abad, C., Safdar, N. The reemergence of measles. *Curr. Infect. Dis. Rep.* 2015; (17): 1–8.

WHO. New Measles Data August 2019 (WHO, 2019).

Momoh, A.A., Ibrahim, M.O., Uwanta, I.J., Manga, S.B. Mathematical model for control of measles epidemiology. Int. J. Pure Appl. Math. 2013;87(5):707–717. doi: 10.12732/ijpam.v87i5.4.

Adewale, S.O., Mohammed, I.T., Olopade, I.A. Mathematical analysis of effect of area on the dynamical spread of measles. IOSR J. Eng. 2014;4(3):43–57. doi: 10.9790/3021-04324357.

Smith, R., Archibald, A., MacCarthy, E., Liu, L., Luke, N.S. A mathematical investigation of vaccination strategies to prevent a measles epidemic. NCJ Math. Stat. 2016;2:29–44.

Garba, S.M., Safi, M.A., Usaini, S. Mathematical model for assessing the impact of vaccination and treatment on measles transmission dynamics. Math. Methods Appl. Sci. 2017;40(18):6371–6388. doi: 10.1002/mma.4462.

Peter, O.J., Afolabi, O.A., Victor, A.A., Akpan, C.E., Oguntolu, F.A. Mathematical model for the control of measles. J. Appl. Sci. Environ. Manag. 2018;22(4):571–576.

Beay, L.K. Modelling the effects of treatment and quarantine on measles, in AIP Conference Proceedings (Vol. 1937, No. 1, p. 020004). AIP Publishing LLC (2018)

Zhang, H., Zhang, J., Zhou, C., Small, M., Wang, B. Hub nodes inhibit the outbreak of epidemicunder voluntary vaccination, New J. Phys. 12, 2010, p. 023015.

Mossong, J., Muller, C.P. Modelling measles re-emergence as a result of waning of immunity in vaccinated populations, Vaccine. 21, 2003, pp. 4597–4603.

Beay, L. K. AIP Conference Proceedings (AIP Publishing LLC, 2004).

Peter, O., Ojo, M. M., Viriyapong, R., Abiodun Oguntolu, F. Mathematical model of measles transmission dynamics using real data from Nigeria. Journal of Difference Equations and Applications, 2022, 28(6), 753-770.

Mickens, R. E. Nonstandard finite difference models of differential equations. world scientific. World Scientific Publishing, Atlanta, 1994, Ga,USA.

Mickens, R. E. Nonstandard finite difference schemes for differential equations. Journal of Difference Equations and Applications, 2002, 8(9), 823-847.

Ongun, M.Y., Turhan, I. A numerical comparison for a discrete HIV infection of CD4+T-Cell model derived from nonstandard numerical scheme. Journal of Applied Mathematics. 2013.4 (2012).

Kocabiyik, M., Ongun, M. Y. Construction a distributed order smoking model and its nonstandard finite difference discretization. AIMS Mathematics, 2022.

Kocabiyik, M., Ongun, M. Y. Discretization and Stability Analysis for a Generalized Type Nonlinear Pharmacokinetic Models. Gazi University Journal of Science , 2023.

Kocabıyık, M., Özdoğan, N., Ongun, M.Y., Nonstandard Finite Difference Scheme for a Computer Virus Model, Journal of Innovative Science and Engineering (JISE),2020, 4(2): 96-108.

Çetinkaya, İ. T., Kocabıyık, M., Ongun, M. Y.Stability Analysis of Discretized Model of Glucose–Insulin Homeostasis. Celal Bayar University Journal of Science, 2021, 17.4: 369-377.

Özdoğan, N., Ongun, M.Y. Dynamical behabiours of a discretized model with Mıchaelıs-Menten Harvesting Rate . Journal of Universal Mathematics, 2022, 5.2: 159-176.

Richter, H. The generalized Henon maps: Examples for higher-dimensional chaos. International Journal of Bifurcation and Chaos, 2002, 12(06), 1371-1384.

CHAPTER IX

Application of Kashuri Fundo Transform to Solve Logistic Growth Model, and Prey-Predator Model

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Introduction

Mathematical modeling is vital for resolving real-life problems in engineering, physics, statistics, economics, finance, chemistry, and other domains. Differential equations often serve as an important tool for modeling problems in these fields. Differential equations comprise quantities whose values fluctuate by one another and their respective rates of change. Modeling the relation between derivatives with differential equations, real-world issues become simpler to comprehend and easier to develop solutions for these

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models. The majority of real-world issues involve nonlinear differential equations. It is challenging to solve some of these problems analytically. As a result, new approaches and studies are being developed to find better and newer numerical solutions to nonlinear equations.

Thomas Malthus introduced one of the earliest mathematical models illustrating the dynamic change of populations. The Malthusian model posits that the rate of population growth in a country is directly proportional to its total population, denoted as $\mathcal{W}(t)$ at any given time t. According to this theory, the population growth at a specific moment in time is directly proportional to the projected population growth in the future. The mathematical representation of this model is described as a first-order ordinary linear differential equation,

$$\frac{d\mathcal{W}}{dt} = \kappa\mathcal{W}$$

with initial condition

$$\mathcal{W}(t_0) = \mathcal{W}_0$$

where, $\mathcal{W}$ represents the population at time t, $\mathcal{W}_0$ indicates the starting population at time t_0 and κ is a constant of proportionality. Based on this equation, it can be deduced that the population graph demonstrates exponential growth (Weigelhofer & Lindsay, 1999).

The Malthus growth model demonstrates that a population can increase unlimitedly over time. Nevertheless, each population has a carrying capacity. When a population achieves its carrying capacity, competition for limited resources such as food, space, and other factors occurs, resulting in a divergence from exponential population growth. As a result, Pierre Verhulst's logistics model, which describes growth in a confined area, replaces the Malthusian model. The nonlinear biological models encompass a logistic growth model within a population, represented by the equation

$$\frac{d\mathcal{W}(t)}{dt} = r\mathcal{W}(t)\left(1 - \frac{\mathcal{W}(t)}{\kappa}\right) \tag{1}$$

where r is growth rate and κ is the carrying capacity. The function $\mathcal{W}(t)$ denotes the population of the species at time t, while the expression $r\mathcal{W}(t)\left(1 - \frac{\mathcal{W}(t)}{\kappa}\right)$ represents the per capita growth rate. Non-dimensionalization of equation (1) is accomplish by

$$\varphi(\tau) = \frac{\mathcal{W}(t)}{\kappa}, \qquad \tau = rt$$

which yields

$$\frac{d\varphi}{d\tau} = \varphi(1 - \varphi). \tag{2}$$

If the initial condition is given as $\mathcal{W}(0) = \mathcal{W}_0$, then $\varphi(0) = \frac{\mathcal{W}_0}{\kappa}$. Consequently, the analytical solution of equation (2) is achieved by

$$\varphi(\tau) = \frac{1}{1 + \left(\frac{\kappa}{\mathcal{W}_0} - 1\right)e^{-\tau}}. \tag{3}$$

The Lotka-Volterra equations were formulated to describe the dynamics of biological systems. This system of nonlinear differential equations focuses on predator-prey interactions. A predator-prey relationship refers to the dynamic between two species and the reciprocal influence they have on each other. At that point, one species is, in fact, consuming the other species for food. A predator is an organism that consumes or hunts other organisms for food, whereas a prey is an organism that another organism kills for food. Examples of predators with their prey are the fox and the rabbit, the lion, and the zebra. Predator-prey dynamics is a notion that applies not just to animals but also to plants. The relationship between the grasshopper and the leaf serves as an illustrative example in this context. The predator-prey models: Lotka–Volterra systems as an interacting species model denoted by

$$\frac{d\mathcal{W}}{dt} = \mathcal{W}(a - b\mathcal{V}) \quad (4)$$

$$\frac{d\mathcal{V}}{dt} = \mathcal{V}(c\mathcal{W} - d) \quad (5)$$

where a, b, c and d are constants (Murray, 1993). Here $\mathcal{W} = \mathcal{W}(t)$ represents the prey population, and $\mathcal{V} = \mathcal{V}(t)$ represents the population of predators at the time t. The non-dimensionalization of the system (4)-(5) is given as

$$\varphi(\tau) = \frac{c\mathcal{W}(t)}{d}, \qquad \psi(\tau) = \frac{b\mathcal{V}(t)}{a}, \qquad \tau = at, \qquad \mu = d/a$$

and it turns into

$$\frac{d\varphi}{d\tau} = \varphi(1 - \psi) \quad (6)$$

$$\frac{d\psi}{d\tau} = \mu[g(\varphi, \psi) - \psi]. \quad (7)$$

Integral transforms are a valuable mathematical tool for solving a wide range of processes and phenomena in the fields of science, engineering, and real-life applications. These transforms allow us to express various complex problems in a mathematical framework, enabling their solution through rigorous mathematical techniques. In fields such as engineering, physics, and chemistry, integral transformations are employed to solve initial value problems, boundary value problems, differential equations, and integral equations. They convert the original domain of problems into another domain, simplifying intricate problems and enhancing comprehensibility. Subsequently, Inverse integral transforms are used to return the solution discovered by integral transforms to its original domain. Due to the diverse range of applications, many novel integral transforms have been defined. The most widely used and well-known integral transforms are the Laplace, Fourier, and Sumudu transforms. In this study, we deal with the Kashuri Fundo transform which is one of the integral transforms. The Kashuri

Fundo transform stands out as a convenient, efficient, easy-to-use, and dependable method, enabling the attainment of solutions without the need for intricate calculations. As a result of this, one may find several studies on the Kashuri Fundo transform in the literature. Kashuri and Fundo (2013) introduced the Kashuri Fundo transform to the literature. Numerous researchers, Kashuri and Fundo among them, developed various applications (Kashuri et al. 2013a, 2013b, 2015; Fundo et al. 2016) of this transform in subsequent stages. Shah et al. (2015a, 2015b) examine to solve a nonlinear differential-difference equation arising in nanotechnology and the concentration of the longitudinal dispersion phenomenon arising in fluid flow through porous media by mixture Kashuri Fundo transform and homotopy perturbation method. Güngör (2021) investigated the solution of Volterra integral equations via Kashuri Fundo transform. Johansyah et al. (2022) solved the economic growth acceleration model with memory effects for the quadratic cost function using Kashuri Fundo transformation method. Moreover, Peker et al. (2022, 2022a, 2022b, 2022c, 2023) utilized this transform to solve Abel's integral equation, steady heat transfer problem, decay problem, cardiovascular models.

Now, we will present the Kashuri Fundo integral transform, along with its necessary properties.

Definition 1. (Kashuri & Fundo, 2013) Let F be a function set defined by

$$F=\left\{f(t)\mid \exists M,k_1,k_2>0,\text{ such that } |f(t)|\le Me^{\frac{|t|}{k_i^2}},\text{ if } t \in (-1)^i\times[0,\infty)\right\}.$$

where M must be finite number k_1, k_2 may be finite or infinite. A new integral transform denoted by the operator $\mathcal{K}(.)$ is defined by

$$\mathcal{K}[f(t)](v)=A(v)=\frac{1}{v}\int_0^\infty e^{\frac{-t}{v^2}}f(t)dt,\ t\ge 0,-k_1<v<k_2.$$

Theorem 1 (Linearity property). (Kashuri & Fundo, 2013) Let $f(t)$ and $g(t)$ be functions whose Kashuri Fundo integral transforms exit and λ_1 , λ_2 be constants, then

$$\mathcal{K}[(\lambda_1 f + \lambda_2\, g)(t)] = \lambda_1\, \mathcal{K}[(f)(t)] + \lambda_2\, \mathcal{K}[(g)(t)].$$

Theorem 2 (Kashuri Fundo transform of the derivative). (Kashuri & Fundo, 2013) Let suppose $A(v)$ be Kashuri Fundo transform of $f(t)$. Then

$$\mathcal{K}\left[f^{(n)}(t)\right] = \frac{A(v)}{v^{2n}} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{v^{2(n-k)-1}}.$$

Table 1. Kashuri Fundo Transform of Some Standart Functions

$f(\mathrm{t})$	$\mathcal{K}[f(\mathrm{t})] = A(v)$
1	v
t	v^3
t^n	$n!\, v^{2n+1}$
$e^{\lambda t}$	$\frac{v}{1-\lambda v^2}$
$\sin(\lambda t)$	$\frac{\lambda v^3}{1+\lambda^2 v^4}$
$\cos(\lambda t)$	$\frac{v}{1+\lambda^2 v^4}$
$\sinh(\lambda t)$	$\frac{\lambda v^3}{1-\lambda^2 v^4}$
$\cosh(\lambda t)$	$\frac{\lambda}{1-\lambda^2 v^4}$

Main Results

This section presents the Kashuri Fundo transform as a technique that facilitates the solution of nonlinear differential equations through biological models, specifically those comprising a logistic growth model to study population dynamics and the prey-predator model to analyze ecological interactions.

Kashuri-Fundo Method for Logistic Growth Model

In this subsection, take into the model equation in the form of

$$\frac{d\varphi}{dt} = \varphi - g(\varphi), \qquad \varphi(0) = \varphi_0 \tag{8}$$

where g represents a nonlinear function of φ. Therefore, we assume that the solution of φ satisfying (8) has a representation of the form of an infinite series

$$\varphi = \varphi(t) = \sum_{k=0}^{\infty} c_k t^k \tag{9}$$

In addition, it fulfills the necessary criteria for the Kashuri Fundo transform to be in existence. Applying the Kashuri Fundo transform to the either side of the differential equation in (8), then we find

$$\frac{\mathcal{K}[\varphi(t)]}{v^2} - \frac{\varphi_0}{v} = \mathcal{K}[\varphi(t)] - \mathcal{K}\big[g\big(\varphi(t)\big)\big].$$

Hence, we obtain

$$\mathcal{K}[\varphi(t)] = \varphi_0 \frac{v}{1 - v^2} - \frac{v^2}{1 - v^2}\mathcal{K}\big[g\big(\varphi(t)\big)\big]. \tag{10}$$

Thus, assuming the existence of the inverse Kashuri Fundo transform $\mathcal{K}^{-1}$ exists and applying it to expression (10), the equation can be written as:

$$\varphi(t)=\varphi_0 e^t-\mathcal{K}^{-1}\left[\frac{v^2}{1-v^2}\mathcal{K}\big[g\big(\varphi(t)\big)\big]\right].$$

Example 1. Let us consider the logistic growth model equation (8) where $\mathcal{W}_0=2$ and $\kappa=1$. Hence φ_0 can be expressed as $\varphi_0=\frac{\mathcal{W}_0}{\kappa}=2$. We set $g(\varphi)=\varphi^2$ as in (2) so that one finds

$$\begin{aligned} g(\varphi) &= \left(\sum_{k=0}^{\infty} c_k t^k\right)^2 \\ &= (c_0+c_1t+c_2t^2+\cdots+c_kt^k+\cdots)^2 \\ &= c_0^2+2c_0c_1t+(2c_0c_2+c_1^2)t^2 \\ &\quad +(2c_0c_3+2c_1c_2)t^3+(2c_0c_4+2c_1c_3+c_2^2)t^4+\cdots. \end{aligned}$$

Implementing Kashuri Fundo transform to each side of the equation

$$\begin{aligned} \mathcal{K}[g(\varphi)] &= G(v) \\ &= c_0^2v+2c_0c_1v^3+(2c_0c_2+c_1^2)2!\,v^5 \\ &\quad +(2c_0c_3+2c_1c_2)3!\,v^7 \\ &\quad +(2c_0c_4+2c_1c_3+c_2^2)4!\,v^9+\cdots. \end{aligned}$$

Using (10) one gets

$$\begin{aligned} \Phi(v) &= 2\,\frac{v}{1-v^2}-\left[\frac{v^3c_0^2}{1-v^2}+\frac{2c_0c_1v^5}{1-v^2}\right. \\ &\quad +2!\,\frac{(2c_0c_2+c_1^2)v^7}{1-v^2}+3!\,\frac{(2c_0c_3+2c_1c_2)v^9}{1-v^2} \\ &\quad \left.+4!\,\frac{(2c_0c_4+2c_1c_3+c_2^2)v^{11}}{1-v^2}+\cdots\right] \\ &= 2\,\frac{v}{1-v^2}-\left[c_0^2\left(\frac{v}{1-v^2}-v\right)+2c_0c_1\left(\frac{v}{1-v^2}-v-v^3\right)\right. \\ &\quad +2!\,(2c_0c_2+c_1^2)\left(\frac{v}{1-v^2}-v-v^3-v^5\right) \end{aligned}$$

$$+3!\,(2c_0c_3+2c_1c_2)\left(\frac{v}{1-v^2}-v-v^3-v^5-v^7\right)$$

$$+4!\,(2c_0c_4+2c_1c_3+c_2^2)\left(\frac{v}{1-v^2}-v-v^3-v^5-v^7-v^9\right)$$

$$+\cdots]$$

Upon application of the inverse Kashuri Fundo transform to this equation yields

$$c_0+c_1t+c_2t^2+c_3t^3+\cdots=2\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\cdots\right)$$

$$-(c_0^2+2c_0c_1+4c_0c_2+2c_1^2+12c_0c_3+12c_1c_2+48c_0c_4+48c_1c_3$$

$$+24c_2^2+\cdots)\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\cdots\right)$$

$$+(c_0^2+2c_0c_1+4c_0c_2+2c_1^2+12c_0c_3+12c_1c_2+48c_0c_4$$

$$+48c_1c_3+24c_2^2+\cdots)$$

$$+(2c_0c_1+4c_0c_2+2c_1^2+12c_0c_3+12c_1c_2+48c_0c_4$$

$$+48c_1c_3+24c_2^2+\cdots)t$$

$$+(2c_0c_1+c_1^2+6c_0c_3+6c_1c_2+24c_0c_4+24c_1c_3+12c_2^2+\cdots)t^2$$

$$+(2c_0c_3+2c_1c_2+8c_0c_4+8c_1c_3+4c_2^2+\cdots)t^3$$

$$+(2c_0c_4+2c_1c_3+c_2^2+\cdots)t^4+\cdots$$

$$=2\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\cdots\right)-c_0^2t-\left(\frac{c_0^2}{2}+c_0c_1\right)t^2$$

$$-\left(\frac{c_0^2}{6}+\frac{c_0c_1}{3}+\frac{2c_0c_2}{3}+\frac{c_1^2}{3}\right)t^3$$

$$-\left(\frac{c_0^2}{24}+\frac{c_0c_1}{12}+\frac{c_0c_2}{6}+\frac{c_1^2}{12}+\frac{c_0c_3}{2}+\frac{c_1c_2}{2}\right)t^4-\cdots$$

$$=2+(2-c_0^2)t+\left(1-\frac{c_0^2}{2}-c_0c_1\right)t^2$$

$$+\left(\frac{1}{3}-\frac{c_0^2}{6}-\frac{c_0c_1}{3}-\frac{2c_0c_2}{3}-\frac{c_1^2}{3}\right)t^3$$

$$+\left(\frac{1}{12}-\frac{c_0^2}{24}-\frac{c_0c_1}{12}-\frac{c_0c_2}{6}-\frac{c_1^2}{12}-\frac{c_0c_3}{2}-\frac{c_1c_2}{2}\right)t^4+\cdots$$

from (9). When the coefficients of power t are equated, the result is

$$c_0=2,$$

$$c_1=2-c_0^2 \Longrightarrow c_1=-2,$$

$$c_2=1-\frac{c_0^2}{2}-c_0c_1 \Longrightarrow c_2=3,$$

$$c_3=\frac{1}{3}-\frac{c_0^2}{6}-\frac{c_0c_1}{3}-\frac{2c_0c_2}{3}-\frac{c_1^2}{3} \Longrightarrow c_3=-\frac{13}{3},$$

$$c_4=\frac{1}{12}-\frac{c_0^2}{24}-\frac{c_0c_1}{12}-\frac{c_0c_2}{6}-\frac{c_1^2}{12}-\frac{c_0c_3}{2}-\frac{c_1c_2}{2} \Longrightarrow c_4=\frac{25}{4},$$

$$\vdots$$

and so on. Consequently, the solution $\varphi(t)$ is obtained from (9) as follows

$$\varphi(t) = 2 - 2t + 3t^2 - \frac{13}{3}t^3 + \frac{25}{4}t^4 + \cdots$$

that is the closed form exact solution obtained in (3). This solution is identical to the one discovered in (Pamuk & Soylu, 2020; Pamuk, 2005).

$s_n(t)$ represents the nth partial sums of the series (9) which is equivalent to

$$s_n(t) = \sum_{k=0}^{n} c_k t^k. \tag{11}$$

Based on the observation of Figure 1, it is evident that a highly accurate approximation of the exact solution for the logistic growth model within the time interval $[0,0.25]$ has been achieved by computing only five terms of the series in (9). This indicates that the rate of convergence of the Kashuri Fundo transform method is highly rapid. Furthermore, it is possible to minimize the overall errors and obtain a reasonably accurate estimation of the exact solution for $t \geq 0.25$ by incorporating new terms into the series.

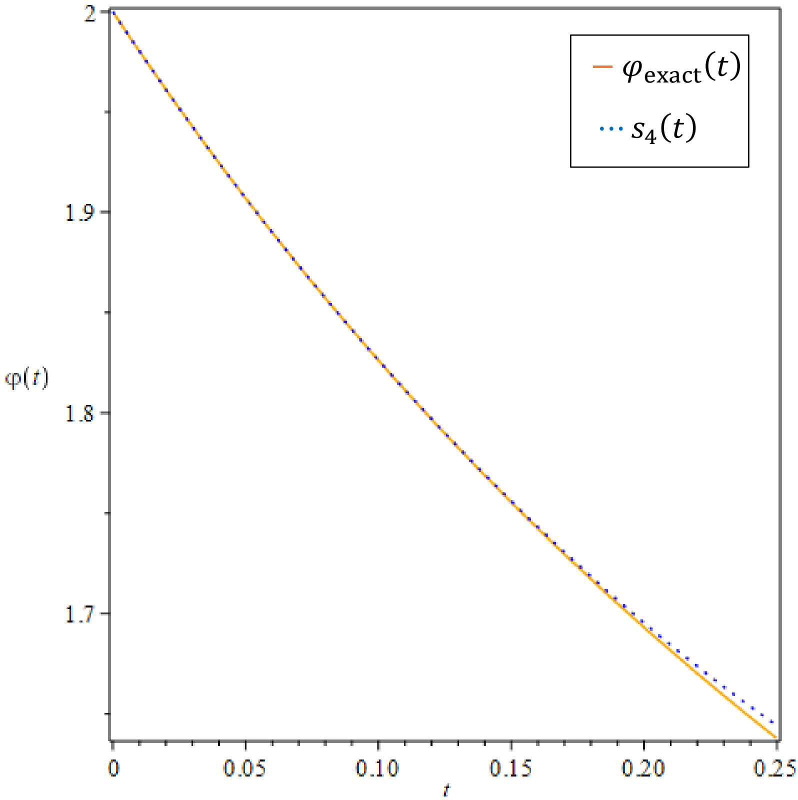

Figure 1. The resolution of the logistic growth model in population dynamics

Applying of Kashuri Fundo Transform for Prey-Predator Model

Let us consider the system of non-linear differential equations that determines the predator-prey model.

$$\frac{d\varphi}{dt} = \varphi - g(\varphi, \psi) \tag{12}$$

$$\frac{d\psi}{dt} = \alpha[h(\varphi, \psi) - \psi] \tag{13}$$

with initial conditions

$$\varphi(0) = \varphi_0, \psi(0) = \psi_0 \tag{14}$$

where g and h are nonlinear functions of φ and ψ and also α be a positive constant. It is assumed that the solutions φ and ψ of the system (12)-(13) possess infinite series expansions in the following form:

$$\varphi(t) = \sum_{k=0}^{\infty} c_k t^k, \qquad \psi(t) = \sum_{k=0}^{\infty} d_k t^k. \tag{15}$$

Furthermore, the necessary criteria for the existence of their Kashuri Fundo transforms are satisfied by them. By utilizing the Kashuri Fundo for the equations (12)-(13) and utilizing (14), we obtain

$$\frac{\Phi(v)}{v^2} - \frac{\varphi_0}{v} = \Phi(v) - G(v) \tag{16}$$

$$\frac{\Psi(v)}{v^2} - \frac{\psi_0}{v} = \alpha[H(v) - \Psi(v)] \tag{17}$$

where $\mathcal{K}[\varphi(t)] = \Phi(v)$, $\mathcal{K}\left[g\left(\varphi(t), \psi(t)\right)\right] = G(v)$, $\mathcal{K}[\psi(t)] = \Psi(v)$, $\mathcal{K}\left[h\left(\varphi(t), \psi(t)\right)\right] = H(v)$ are the Kashuri Fundo transforms of the functions $\varphi(t)$, $g\left(\varphi(t), \psi(t)\right)$, $\psi(t)$ and

$h(\varphi(t), \psi(t))$, respectively. By solving the equations (16)-(17) for $\Phi(v)$ and $\Psi(v)$, one gets

$$\Phi(v) = \frac{v}{1 - v^2}\varphi_0 - \frac{v^2}{1 - v^2} G(v)$$

$$\Psi(v) = \frac{v}{1 + \alpha v^2}\psi_0 + \alpha \frac{v^2}{1 + \alpha v^2} H(v)$$

Assuming inverse Kashuri Fundo transforms exist and utilizing them to the system, we obtain

$$\varphi(t) = \varphi_0 e^t - \mathcal{K}^{-1}\left[\frac{v^2}{1 - v^2} G(v)\right]$$

$$\psi(t) = \psi_0 e^{-\alpha t} + \alpha\mathcal{K}^{-1}\left[\frac{v^2}{1 + \alpha v^2} H(v)\right]$$

desired solutions to the initial value problem (12)-(14).

Example 2. Consider the differential equation system that governs the predator and prey model

$$\frac{d\varphi}{dt} = \varphi - \varphi\psi \tag{18}$$

$$\frac{d\psi}{dt} = \varphi\psi - \psi \tag{19}$$

with initial data $\varphi(0) = 1.3,\ \psi(0) = 0.6$.

Suppose that $\varphi(t) = \sum_{k=0}^{\infty} c_k t^k,\ \psi(t) = \sum_{k=0}^{\infty} d_k t^k$ be solutions of the system of (18)-(19). Hence, we find

$$g(\varphi,\psi) = h(\varphi,\psi) = \varphi\psi = \left(\sum_{k=0}^{\infty} c_k t^k\right)\left(\sum_{k=0}^{\infty} d_k t^k\right)$$

$$= c_0 d_0 + (c_0 d_1 + c_1 d_0)t + (c_0 d_2 + c_1 d_1 + c_2 d_0)t^2$$

$$+(c_0 d_3 + c_1 d_2 + c_2 d_1 + c_3 d_0)t^3 + \cdots.$$

The corresponding Kashuri Fundo transforms of these functions become

$$G(v) = H(v) = \mathcal{K}[\varphi\psi]$$

$$= c_0 d_0 \mathcal{K}[1] + (c_0 d_1 + c_1 d_0)\mathcal{K}[t]$$

$$+(c_0 d_2 + c_1 d_1 + c_2 d_0)\mathcal{K}[t^2]$$

$$+(c_0 d_3 + c_1 d_2 + c_2 d_1 + c_3 d_0)\mathcal{K}[t^3] + \cdots$$

$$= c_0 d_0 v + (c_0 d_1 + c_1 d_0) v^3$$

$$+(c_0 d_2 + c_1 d_1 + c_2 d_0) 2!\, v^5$$

$$+(c_0 d_3 + c_1 d_2 + c_2 d_1 + c_3 d_0) 3!\, v^7 + \cdots.$$

Therefore, we find

$$\Phi(v) = \frac{v}{1-v^2} 1.3 - \frac{v^3}{1-v^2} c_0 d_0 - \frac{v^5}{1-v^2}(c_0 d_1 + c_1 d_0)$$

$$-\frac{2!\, v^7}{1-v^2}(c_0 d_2 + c_1 d_1 + c_2 d_0)$$

$$-\frac{3!\, v^9}{1-v^2}(c_0 d_3 + c_1 d_2 + c_2 d_1 + c_3 d_0) - \cdots$$

$$= \frac{v}{1-v^2} 1.3 - \left(\frac{v}{1-v^2} - v\right) c_0 d_0$$

$$-\left(\frac{v}{1-v^2}-v-v^3\right)(c_0d_1+c_1d_0)$$

$$-2!\left(\frac{v}{1-v^2}-v-v^3-v^5\right)(c_0d_2+c_1d_1+c_2d_0)$$

$$-3!\left(\frac{v}{1-v^2}-v-v^3-v^5-v^7\right)(c_0d_3+c_1d_2$$

$$+c_2d_1+c_3d_0)-\cdots$$

and

$$\Psi(v)=\frac{v}{1+v^2}0.6+\frac{v^3}{1+v^2}c_0d_0+\frac{v^5}{1+v^2}(c_0d_1+c_1d_0)$$

$$+\frac{2!\,v^7}{1+v^2}(c_0d_2+c_1d_1+c_2d_0)$$

$$+\frac{3!\,v^9}{1+v^2}(c_0d_3+c_1d_2+c_2d_1+c_3d_0)+\cdots$$

$$=\frac{v}{1+v^2}0.6+\left(v-\frac{v}{1+v^2}\right)c_0d_0$$

$$+\left(v^3-v+\frac{v}{1+v^2}\right)(c_0d_1+c_1d_0)$$

$$+2!\left(v^5-v^3+v-\frac{v}{1+v^2}\right)(c_0d_2+c_1d_1+c_2d_0)$$

$$+3!\left(v^7-v^5+v^3-v+\frac{v}{1-v^2}\right)(c_0d_3+c_1d_2$$

$$+c_2d_1+c_3d_0)+\cdots$$

by using (15). By utilizing inverse Kashuri Fundo transform to these equations, we obtain

$$c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots$$

$$= 1.3\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) - c_0 d_0 t$$

$$-(c_0 d_0 + c_0 d_1 + c_1 d_0)\frac{t^2}{2!}$$

$$-(c_0 d_0 + c_0 d_1 + c_1 d_0 + 2c_0 d_2 + 2c_1 d_1 + 2c_2 d_0)\frac{t^3}{3!} - \cdots$$

$$= 1.3 + (1.3 - c_0 d_0)t + (1.3 - c_0 d_0 - c_0 d_1 - c_1 d_0)\frac{t^2}{2!}$$

$$+(1.3 - c_0 d_0 - c_0 d_1 - c_1 d_0 - 2c_0 d_2 - 2c_1 d_1 - 2c_2 d_0)\frac{t^3}{3!}$$

$$+\cdots$$

and

$$d_0 + d_1 t + d_2 t^2 + d_3 t^3 + \cdots$$

$$= 0.6\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right)$$

$$-c_0 d_0 t - (c_0 d_0 + c_0 d_1 + c_1 d_0)\frac{t^2}{2!}$$

$$-(c_0 d_0 + c_0 d_1 + c_1 d_0 + 2c_0 d_2 + 2c_1 d_1 + 2c_2 d_0)\frac{t^3}{3!} - \cdots$$

$$= 0.6 + (c_0 d_0 - 0.6)t + (0.6 - c_0 d_0 + c_0 d_1 + c_1 d_0)\frac{t^2}{2!}$$

$$+(-0.6 + c_0 d_0 - c_0 d_1 - c_1 d_0 + 2c_0 d_2 + 2c_1 d_1 + 2c_2 d_0)\frac{t^3}{3!} + \cdots \ .$$

If the coefficients are equalized to powers of t, it is found as

$$c_0 = 1.3 \qquad d_0 = 0.6$$

$$c_1 = 1.3 - c_0 d_0 \qquad d_1 = c_0 d_0 - 0.6$$
$$c_1 = 0.52 \qquad d_1 = 0.18$$

$$c_2 = \frac{1}{2!}(1.3 - c_0 d_0 - c_0 d_1 - c_1 d_0) \qquad d_2 = \frac{1}{2!}(0.6 + c_0 d_1 + c_1 d_0 - c_0 d_0)$$

$$c_2 = -0.013 \qquad d_2 = 0.183$$

$$c_3 = \frac{1}{3!}(1.3 - c_0 d_0 - c_0 d_1 - c_1 d_0 - 2c_0 d_2 - 2c_1 d - 2c_2 d_0) \qquad d_3 = \frac{1}{3!}(-0.6 + c_0 d_0 - c_0 d_1 - c_1 d_0 + 2c_0 d_2 + 2c_1 d_1 + 2c_2 d_0)$$

$$c_4 = -0.1122 \qquad d_4 = 0.0469$$
$$\vdots \qquad \vdots$$

The subsequent terms of the series can be obtained using this method. By substituting these terms into equation (15), we obtain the approximate solutions for the problem described by equations (18)-(19):

$$\varphi(t) = 1.3 + 0.52t - 0.013t^2 - 0.1122t^3 - \cdots$$

$$\psi(t) = 0.6 + 0.18t + 0.183t^2 + 0.0469t^3 + \cdots \ .$$

The current results we have obtained align with the findings from the referenced research. in reference (Pamuk & Soylu, 2020; Pamuk, 2005).

The approximate solutions to systems (18)-(19) derived through the Kashuri Fundo transform using solely four elements of the series (15) are illustrated in Figure 2. The numerical solutions for this system appear in Figure 3. The system's numerical solutions are computed with Ode45, an integrated ordinary differential equation solver in MATLAB.

Within the time period of [0,1.5], the two solutions for φ (prey population) and ψ (predator population) are found to be quite near when comparing the two figures. Adding more terms to the series provides an even closer approximation to the numerical answer for $t \geq 1.5$, as previously mentioned in the context of the logistic growth model.

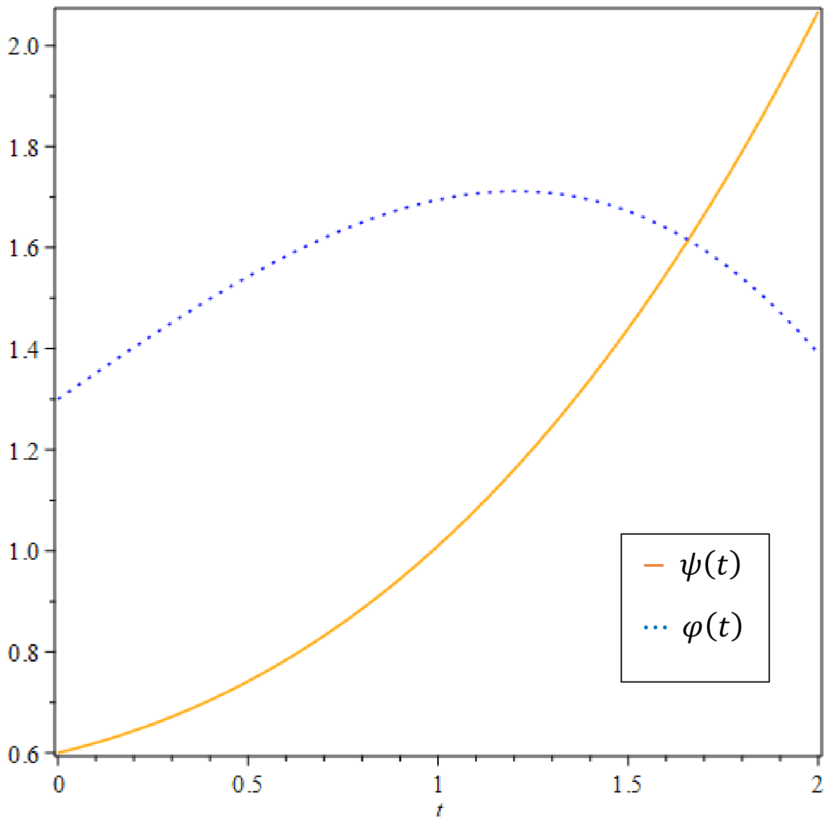

Figure 2. Approximate solutions to the system (18)-(19) by Kashuri Fundo transform method

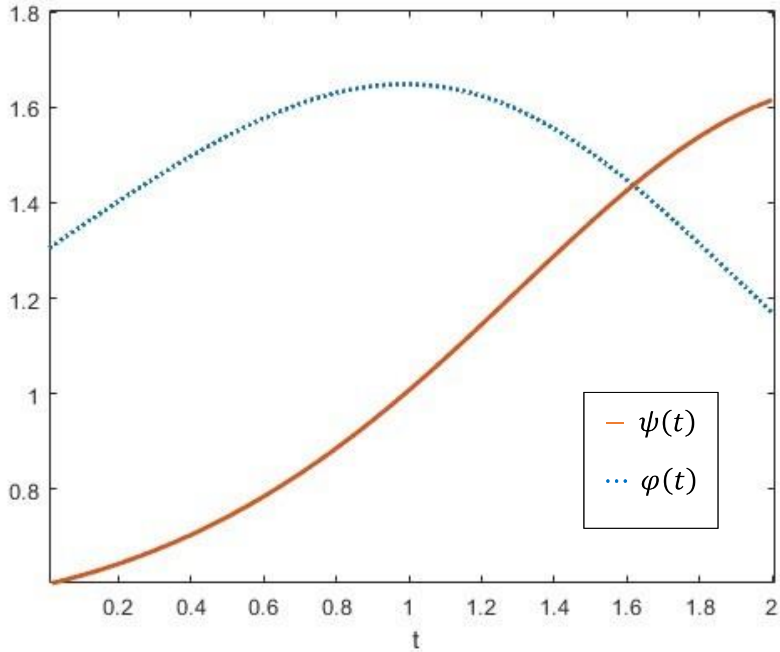

Figure 3. Numerical solutions to the system (18)-(19)

REFERENCES

Cuha, F. A. & Peker, H. A. (2022) Solution of Abel's Integral Equation by Kashuri Fundo Transform. *Thermal Science*, 26(4A), 3003-3010

Fundo, A., Kashuri, A. & Liko R. (2016) New Integral Transform in Caputo Type Fractional Difference Operator. *Universal Journal of Applied Science*, 4(1), 7-10.

Güngör, N. (2021) Solving Convolution Type Linear Volterra Integral Equations with Kashuri Fundo Transform. *Journal of Abstract and Computational Mathematics*, 6 (2), 1–7.

Johansyah, M. D., Supriatna, A. K., Rusyaman, E. & Saputra, J. (2022) Solving the economic growth acceleration model with memory effects: an application of combined theorem of Adomian decomposition methods and Kashuri-Fundo transformation methods. *Symmetry*, 14 (2) 192.

Kashuri, A. & Fundo, A. (2013) A New Integral Transform. *Advances in Theoretical and Applied Mathematics*, 8(1), 27-43.

Kashuri, A., Fundo, A. & Liko, R., (2013a) On Double New Integral Transform and Double Laplace Transform. *European Scientific Journal*, 9(33), 1857–7881.

Kashuri, A., Fundo, A. & Kreku M. (2013b) Mixture of a New Integral Transform and Homotopy Perturbation Method for Solving Nonlinear Partial Differential Equations. *Advances in Pure Mathematics*, 3, 317-323.

Kashuri, A., Fundo A. & Liko, R. (2015) New Integral Transform For Solving Some Fractional Differential Equations. *International Journal of Pure and Applied Mathematics*, 103(4), 675-682.

Murray, J. D. (1993) *Mathematical Biology*, Springer, Berlin.

Pamuk, S. (2005) The decomposition method for continuous population models for single and interacting species. *Applied Mathematics and Computation*, 163, 79-88.

Pamuk, S. & Soylu N. (2020) Laplace transform method for logistic growth in a population and predator models. *New Trends in Mathematical Sciences*, 8(3), 9-17.

Peker, H. A., Cuha, F. A. & Peker, B. (2022a) Solving Steady Heat Transfer Problems via Kashuri Fundo Transform. *Thermal Science*, 26(4A),3011-3017.

Peker, H. A. & Cuha, F. A. (2022b) Application of Kashuri Fundo Transform and Homotopy Perturbation Methods to Fractional Heat Transfer and Porous Media Equations. *Thermal Science*, 26(4A), 2877-2884.

Peker, H. A. & Cuha, F. A. (2022c) Application of Kashuri Fundo Transform to Decay Problem. *SDU Journal of Natural and Applied Sciences*, 26 (3), 546–551.

Peker, H. A. & Cuha, F. A. (2023) Exact Solutions of Some Basic Cardiovascular Models by Kashuri Fundo Transform. Journal of New Theory, 43, 63-72.

Shah, K. & Singh, T. (2015a) A Solution of the Burger's Equation Arising in the Longitudinal Dispersion Phenomenon in Fluid Flow through Porous Media by Mixture of New Integral Transform and Homotopy Perturbation Method. *Journal of Geoscience and Environment Protection*, 3, 24-30.

Shah K. & Singh T. (2015b) The Mixture of New Integral Transform and Homotopy Perturbation Method for Solving Discontinued Problems Arising in Nanotechnology. *Open Journal of Applied Sciences*, 5, 688-695.

Weigelhofer, W.S. & Lindsay, K.A. (1999) *Ordinary differential equations & applications: Mathematical methods for applied mathematicians, physicists, engineers and bioscientists*, Woodhead.

CHAPTER X

The Representations and Fnite Sums of the Mersenne-Padovan Numbers

Özgür ERDAĞ

Introduction and Preliminaries

A Mersenne number, by M_n , is a number of the form $M_n = 2^n - 1$. The Mersenne sequence $\{M_n\}_{n \geq 0}$ can also be defined recursively by

$$M_{n+2} = 3M_{n+1} - 2M_n$$

with initial values $M_0 = 0$ and $M_1 = 1$. It is worth noting that Mersenne numbers belong to the same family as Fermat numbers, and thus, they share the same properties. (Catarino, Campos & Vasco, 2016)

The Padovan sequence is the sequence of the integer $\{P(n)\}$ defined by the following recurrence relation:

$$P(n) = P(n-2) + P(n-3)$$

for $n \geq 3$ and with initial values $P(0) = P(1) = P(2) = 1$.

It is easy to see that the characteristic polynomials of the Mersenne sequence and Padovan sequence are $k_1(x) = x^2 - 3x + 2$ and $k_2(x) = x^3 - x - 1$, respectively.

Erdağ (Erdağ, 2023) defined the Mersenne-Padovan sequence $\{M_n^{Pa}\}$ by the following homogeneous linear recurrence relation:

$$M_{n+5}^{Pa} = 3M_{n+4}^{Pa} - M_{n+3}^{Pa} - 2M_{n+2}^{Pa} - M_{n+1}^{Pa} + 2M_n^{Pa} \quad (1)$$

for $n \geq 0$ and with initial conditions $M_0^{Pa} = \cdots = M_3^{Pa} = 0$ and $M_4^{Pa} = 1$.

Also in (Erdağ, 2023), by the recurrence relation (1), they have

$$\begin{bmatrix} M_{n+5}^{Pa} \\ M_{n+4}^{Pa} \\ M_{n+3}^{Pa} \\ M_{n+2}^{Pa} \\ M_{n+1}^{Pa} \end{bmatrix} = \begin{bmatrix} 3 & -1 & -2 & -1 & 2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} M_{n+4}^{Pa} \\ M_{n+3}^{Pa} \\ M_{n+2}^{Pa} \\ M_{n+1}^{Pa} \\ M_n^{Pa} \end{bmatrix}$$

for the Mersenne-Padovan sequence $\{M_n^{Pa}\}$ and they gave the generating matrix of the Mersenne-Padovan sequence $\{M_n^{Pa}\}$ as follows:

$$E_{Pa}^{M} = \begin{bmatrix} 3 & -1 & -2 & -1 & 2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The matrix $E_{Pa}^{M} = \left[e_{ij} \right]_{5\times5}$ is said to be the Mersenne-Padovan matrix. Then, they obtained that

$$\left(E_{Pa}^{M}\right)^{n} = \begin{bmatrix} M_{n+4}^{Pa} & -M_{n+4}^{Pa} + P(n+4) - 1 & -M_{n+4}^{Pa} + P(n) & -P(n+2) + 1 & 2M_{n+3}^{Pa} \\ M_{n+3}^{Pa} & -M_{n+3}^{Pa} + P(n+3) - 1 & -M_{n+3}^{Pa} + P(n-1) & -P(n+1) + 1 & 2M_{n+2}^{Pa} \\ M_{n+2}^{Pa} & -M_{n+2}^{Pa} + P(n+2) - 1 & -M_{n+2}^{Pa} + P(n-2) & -P(n) + 1 & 2M_{n+1}^{Pa} \\ M_{n+1}^{Pa} & -M_{n+1}^{Pa} + P(n+1) - 1 & -M_{n+1}^{Pa} + P(n-3) & -P(n-1) + 1 & 2M_{n}^{Pa} \\ M_{n}^{Pa} & -M_{n}^{Pa} + P(n) - 1 & -M_{n}^{Pa} + P(n-4) & -P(n-2) + 1 & 2M_{n-1}^{Pa} \end{bmatrix}.$$

for $n \geq 4$, which can be readily established by mathematical induction.

Many authors have recently investigated the characteristics of number theory, such as those derived from homogeneous linear recurrence relations, which are pertinent to this study; see for example: (Bradie, 2010; Horadam, 1994; Shannon, Horadam & Anderson, 2006; Taşçı & Firengiz, 2010; Tuğlu, Koçer & Stakhov, 2011). In (Aküzüm, 2020; Aküzüm & Deveci, 2021; Deveci & Aküzüm, 2022; Deveci, Aküzüm & Rashedi, 2022; Kilic, 2008; Kılıc, 2009; Kilic & Tasci, 2006; Stakhov, 1999), the authors defined some linear recurrence sequences and provided their various properties using matrix methods. Obtaining new sequences by multiplication of the characteristic polynomials of the sequences was first started in (Deveci, 2021). Later the concept was expanded by authors to different linear recurrence sequences; see for example: (Deveci & Shannon, 2021; Erdağ & Deveci, 2020a, 2020b; Erdağ & Deveci, 2021; Erdağ & Deveci, 2022; Erdağ, Deveci & Shannon, 2020; Shannon, Erdağ & Deveci, 2021). In this study, we consider the Mersenne-Padovan sequence. Also, we derive the permanental and the determinantal representations of the Mersenne-Padovan numbers by using certain matrices that are obtained from the generating matrix of the Mersenne-Padovan sequence. Finally, we obtain the combinatorial and exponential representations and the finite sums of the Mersenne-Padovan numbers by the aid of the generating function and the generating matrix of the Mersenne-Padovan sequence.

Main Results

Definition 1. A $u \times v$ real matrix $M = \left[m_{i,j} \right]$ is called a contractible matrix in the kth column (resp. row.) if the kth column (resp. row.) contains exactly two non-zero entries.

Suppose that $x_1, x_2, \ldots, x_u$ are row vectors of the matrix M. If M is contractible in the kth column such that $m_{i,k} \neq 0, m_{j,k} \neq 0$ and $i \neq j$, then the $(u-1)\times(v-1)$ matrix $M_{ij:k}$ is obtained from M by replacing the ith row with $m_{i,k} x_j + m_{j,k} x_i$ and deleting the jth row. The kth column is called the contraction in the kth column relative to the ith row and the jth row.

In (Brualdi & Gibson, 1997), Brualdi and Gibson obtained that $per(M) = per(N)$ if M is a real matrix of order $\alpha > 1$ and N is a contraction of M.

Now we concentrate on finding relationships among the Mersenne-Padovan numbers and the permanents of certain matrices which are obtained by using the generating matrix of the Mersenne-Padovan sequences.

Let $F_\eta^{M,Pa} = \left[f_{ij} \right]$ be the $\eta \times \eta$ super-diagonal matrix, defined by

$$f_{ij} = \begin{cases} 3, & \text{if } i=\tau \text{ and } j=\tau \text{ for } 1\le\tau\le\eta, \\ 2, & \text{if } i=\tau \text{ and } j=\tau+4 \text{ for } 1\le\tau\le\eta-4, \\ 1, & \text{if } i=\tau+1 \text{ and } j=\tau \text{ for } 1\le\tau\le\eta-1, \\ -1, & \begin{array}{c} \text{if } i=\tau \text{ and } j=\tau+1 \text{ for } 1\le\tau\le\eta-1 \\ \text{and} \\ i=\tau \text{ and } j=\tau+3 \text{ for } 1\le\tau\le\eta-3, \end{array} \\ -2, & \text{if } i=\tau \text{ and } j=\tau+2 \text{ for } 1\le\tau\le\eta-2, \\ 0, & \text{otherwise.} \end{cases}$$

for $\eta \ge 5$. Then we have the following Theorem.

Theorem 1. For $\eta \ge 5$,

$$perF_{\eta}^{M,Pa} = M_{\eta+4}^{Pa}.$$

Proof. Let us consider matrix $F_{\eta}^{M,Pa}$ and the equation be hold for $\eta \ge 5$. Then we show that the equation holds for $\eta+1$. If we expand the $perF_{\eta}^{M,Pa}$ by the Laplace expansion of permanent with respect to the first row, then we obtain

$$perF_{\eta+1}^{M,Pa} = 3perF_{\eta}^{M,Pa} - perF_{\eta-1}^{M,Pa} - 2perF_{\eta-2}^{M,Pa} - perF_{\eta-3}^{M,Pa} + 2perF_{\eta-4}^{M,Pa}.$$

Since

$$perF_{\eta}^{M,Pa} = M_{\eta+4}^{Pa},$$

$$perF_{\eta-1}^{M,Pa} = M_{\eta+3}^{Pa},$$

$$perF_{\eta-2}^{M,Pa} = M_{\eta+2}^{Pa},$$

$$perF_{\eta-3}^{M,Pa} = M_{\eta+1}^{Pa}$$

and

$$perF_{\eta-4}^{M,Pa} = M_{\eta}^{Pa}$$

we easily obtain that $perF_{\eta+1}^{M,Pa} = M_{\eta+5}^{Pa}$. So the proof is complete.

Let $G_{\eta}^{M,Pa} = \left[g_{ij} \right]$ be the $\eta \times \eta$ matrix, defined by

$$g_{ij} = \begin{cases} 3, & \text{if } i = \tau \text{ and } j = \tau \text{ for } 1 \le \tau \le \eta - 1, \\ 2, & \text{if } i = \tau \text{ and } j = \tau + 4 \text{ for } 1 \le \tau \le \eta - 4, \\ & \text{if } i = \tau + 1 \text{ and } j = \tau \text{ for } 1 \le \tau \le \eta - 2 \\ 1, & \text{and} \\ & i = \tau \text{ and } j = \tau \text{ for } \tau = \eta, \\ & \text{if } i = \tau \text{ and } j = \tau + 1 \text{ for } 1 \le \tau \le \eta - 2 \\ -1, & \text{and} \\ & i = \tau \text{ and } j = \tau + 3 \text{ for } 1 \le \tau \le \eta - 3, \\ -2, & \text{if } i = \tau \text{ and } j = \tau + 2 \text{ for } 1 \le \tau \le \eta - 3, \\ 0, & \text{otherwise.} \end{cases}$$

for $\eta \ge 5$. Then we have the following Theorem.

Theorem 2. For $\eta \ge 5$,

$$perG_{\eta}^{M,Pa} = M_{\eta+3}^{Pa}.$$

Proof. Let us consider matrix $G_{\eta}^{M,Pa}$ and the equation be hold for $\eta \ge 5$. Then we show that the equation holds for $\eta + 1$. If we expand the $perG_{\eta}^{M,Pa}$ by the Laplace expansion of permanent with respect to the first row, then we obtain

$$perG_{\eta+1}^{M,Pa} = 3perG_{\eta}^{M,Pa} - perG_{\eta-1}^{M,Pa} - 2perG_{\eta-2}^{M,Pa} - perG_{\eta-3}^{M,Pa} + 2perG_{\eta-4}^{M,Pa}$$

.

Since

$$perG_{\eta}^{M,Pa} = M_{\eta+3}^{Pa},$$

$$perG_{\eta-1}^{M,Pa} = M_{\eta+2}^{Pa},$$

$$perG_{\eta-2}^{M,Pa} = M_{\eta+1}^{Pa},$$

$$perG_{\eta-3}^{M,Pa} = M_{\eta}^{Pa}$$

and

$$perG_{\eta-4}^{M,Pa} = M_{\eta-1}^{Pa}$$

we easily obtain that $perG_{\eta+1}^{M,Pa} = M_{\eta+4}^{Pa}$. So the proof is complete.

Assume that $H_{\eta}^{M,Pa} = \left[h_{ij}\right]$ is the $\eta \times \eta$ matrix, defined by

$$H_{\eta}^{M,Pa} = \begin{bmatrix} 1 & \cdots & \overset{\substack{(\eta-1)\text{th} \\ \downarrow}}{1} & 0 \\ 1 & & & \\ 0 & & G_{\eta-1}^{M,Pa} & \\ \vdots & & & \\ 0 & & & \end{bmatrix}$$

for $\eta > 5$, then we have the following results:

Theorem 3. For $\eta > 5$,

$$perH_{\eta}^{M,Pa} = \sum_{i=0}^{\eta+2} M_{i}^{Pa}.$$

Proof. If we extend $perG_{\alpha}^{M,F}$ with respect to the first row, we write

$$perH_{\eta}^{M,Pa} = perH_{\eta-1}^{M,Pa} + perG_{\eta-1}^{M,Pa}$$

Thus, by the results and an inductive argument, the proof is easily seen. □

A matrix K is called convertible if there is an $n \times n$ $(1,-1)$-matrix L such that $perK = \det(K * L)$, where $K * L$ denotes the Hadamard product of K and L.

Now we give relationships among the Mersenne-Padovan numbers and the determinants of certain matrices which are obtained by using the matrices $F_\eta^{M,Pa}$, $G_\eta^{M,Pa}$ and $H_\eta^{M,Pa}$. Let $\eta > 5$ and let R be the $\eta \times \eta$ matrix, defined by

$$R = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & -1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & -1 & 1 \end{bmatrix}.$$

Corollary 1. For $\eta > 5$,

$$\det\left(F_\eta^{M,Pa} * R\right) = M_{\eta+4}^{Pa},$$

$$\det\left(G_\eta^{M,Pa} * R\right) = M_{\eta+3}^{Pa}$$

and

$$\det\left(H_\eta^{M,Pa} * R\right) = \sum_{i=0}^{\eta+2} M_i^{Pa} .$$

Proof. Since $perF_{\eta}^{M,Pa}=\det\left(F_{\eta}^{M,Pa}*R\right)$, $perG_{\eta}^{M,Pa}=\det\left(G_{\eta}^{M,Pa}*R\right)$ and $perH_{\eta}^{M,Pa}=\det\left(H_{\eta}^{M,Pa}*R\right)$ for $\eta>5$ by Theorem 1., Theorem 2. and Theorem 3., we have the conclusion.
□

Let $K\left(k_1,k_2,\dots,k_v\right)$ be a $v\times v$ companion matrix as follows:

$$K\left(k_1,k_2,\dots,k_v\right)=\begin{bmatrix} k_1 & k_2 & \cdots & k_v \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}.$$

For more details on the companion type matrices, see (Lancaster & Tismenetsky, 1985; Lidl & Niederreiter, 1986).

Theorem 4. (Chen and Louck (Chen & Louck, 1996)) The (i,j) entry $k_{i,j}^{(n)}\left(k_1,k_2,\dots,k_v\right)$ in the matrix $K^n\left(k_1,k_2,\dots,k_v\right)$ is given by the following formula:

$$k_{i,j}^{(n)}\left(k_1,k_2,\dots,k_v\right)=\sum_{\left(t_1,t_2,\dots,t_v\right)}\frac{t_j+t_{j+1}+\cdots+t_v}{t_1+t_2+\cdots+t_v}\times\binom{t_1+\cdots+t_v}{t_1,\dots,t_v}k_1^{t_1}\cdots k_v^{t_v}$$

(2)

where the summation is over nonnegative integers satisfying $t_1+2t_2+\cdots+vt_v=n-i+j$, $\binom{t_1+\cdots+t_v}{t_1,\dots,t_v}=\frac{\left(t_1+\cdots+t_v\right)!}{t_1!\cdots t_v!}$ is a multinomial coefficient, and the coefficients in (2) are defined to be 1 if $n=i-j$.

Then we can give other combinatorial representations than for the Mersenne-Padovan numbers by the following Corollary.

Corollary 2. Let M_n^{Pa} be the nth Mersenne-Padovan number for $n \geq 4$. Then

i.

$$M_n^{Pa} = \sum_{(t_1,t_2,t_3,t_4,t_5)} \binom{t_1+t_2+t_3+t_4+t_5}{t_1,t_2,t_3,t_4,t_5} 3^{t_1} 2^{t_5} (-1)^{t_2+t_4} (-2)^{t_3}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + 3t_3 + 4t_4 + 5t_5 = n-4$.

ii.

$$M_n^{F} = \frac{1}{2} \sum_{(t_1,t_2,t_3,t_4,t_5)} \frac{t_5}{t_1+t_2+t_3+t_4+t_5} \times \binom{t_1+t_2+t_3+t_4+t_5}{t_1,t_2,t_3,t_4,t_5} 3^{t_1} 2^{t_5} (-1)^{t_2+t_4} (-2)^{t_3}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + 3t_3 + 4t_4 + 5t_5 = n+1$.

Proof. If we take $i=5$, $j=1$ for the case i. and $i=4$, $j=5$ for the case ii. in Theorem 4., then we can directly see the conclusions from $\left(E_{Pa}^{M}\right)^n$. □

It is easy to see that the generating function of the Mersenne-Padovan sequence $\left\{M_n^{Pa}\right\}$ is as follows:

$$q(x) = \frac{x^4}{1-3x+x^2+2x^3+x^4-2x^5}, \left(0 \leq 3x - x^2 - 2x^3 - x^4 + 2x^5 < 1\right)$$

.

Now considering the function $q(x)$, we can give an exponential representation for the Mersenne-Padovan sequence by the following Theorem.

Theorem 5. Let $q(x)$ be generating function for the Mersenne-Padovan sequence. The exponential representation for the Mersenne-Padovan sequence is as follows:

$$q(x) = x^4 \exp\left(\sum_{i=1}^{\infty} \frac{(x)^i}{i} \left(3 - x - 2x^2 - x^3 + 2x^4\right)^i \right).$$

Proof. Since

$$\ln \frac{q(x)}{x^4} = -\ln\left(1 - 3x + x^2 + 2x^3 + x^4 - 2x^5\right)$$

and

$$\begin{aligned} \ln\left(1 - 3x + x^2 + 2x^3 + x^4 - 2x^5\right) = -\Big[& x\left(3 - x - 2x^2 - x^3 + 2x^4\right) + \\ & \frac{1}{2} x^2 \left(3 - x - 2x^2 - x^3 + 2x^4\right)^2 + \\ & \cdots + \frac{1}{i} x^i \left(3 - x - 2x^2 - x^3 + 2x^4\right)^i \end{aligned}$$

by a simple calculation, we obtain the conclusion. □

Now we consider the sums of the Mersenne-Padovan numbers. Let

$$S_n = \sum_{j=0}^{n} M_j^{Pa}$$

for $n \geq 4$ and let Q_{Pa}^M and $\left(Q_{Pa}^M\right)^n$ be the 6×6 matrix such that

$$Q_{Pa}^{M}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & & & & \\ 0 & & E_{Pa}^{M} & & \\ 0 & & & & \\ 0 & & & & \end{bmatrix}.$$

If we use induction on n, then we obtain

$$\left(Q_{Pa}^{M}\right)^{n}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ S_{n+3} & & & & & \\ S_{n+2} & & & & & \\ S_{n+1} & & & \left(E_{Pa}^{M}\right)^{n} & & \\ S_{n} & & & & & \\ S_{n-1} & & & & & \end{bmatrix}.$$

References

Aküzüm, Y. (2020). The Hadamard-type Padovan-*p* sequences. *Turkish Journal of Science*, *5* (2), 102-109.

Aküzüm, Y. & Deveci, Ö. (2021). The Arrowhead-Jacobsthal Sequences. *Mathematica Montisnigri*, *51* (3), 31-44.

Bradie, B. (2010). Extension and refinements of some properties of sums involving Pell number. *Missouri Journal of Mathematical Sciences*, *22* (1), 37-43.

Brualdi, R. A. & Gibson, P. B. (1997). Convex polyhedra of doubly stochastic matrices. I: applications of permanent function. *Journal of Combinatorial Theory*, *Series A*, *22* (2), 194-230.

Catarino, P., Campos, H. & Vasco, P. (2016). On the Mersenne sequence. *Annales Mathematicae et Informaticae*, *46*, 37-53.

Chen, W. Y. & Louck, J. D. (1996). The combinatorial power of the companion matrix. *Linear Algebra and its Applications*, *232*, 261-278.

Deveci, Ö. & Aküzüm, Y. (2022). The Hadamard-type *k*-step Fibonacci sequences. *Analele Stiintifice ale Universitatii Al. I. Cuza din Iasi-Matematica*, *68* (f.2), 153-165.

Deveci, Ö., Aküzüm, Y. & Rashedi, M. E. (2022). The Hadamard-type *k*-step Pell sequences. *Notes on Number Theory and Discrete Mathematics*, *28* (2), 339-349.

Deveci, Ö. (2021). On the connections among Fibonacci, Pell, Jacobsthal and Padovan numbers. *Notes on Number Theory and Discrete Mathematics*, *27* (2), 111-128.

Deveci, Ö. & Shannon A. G. (2021). Matrix Manipulations for Properties of Fibonacci *p*-Numbers and their Generalizations. *Analele Stiintifice ale Universitatii Al. I. Cuza din Iasi-Matematica*, *67* (f.1), 85-95.

Erdağ, Ö. (2023). The Mersenne-Padovan Sequence and Binet Formulas. 8. International Sciences and Innovation Congress, 14-15 October 2023, Ankara, Turkey, Congress Book, (pp. 262-270).

Erdağ, Ö. & Deveci, Ö. (2020a). On The Connections Between Jacobsthal Numbers and Fibonacci *p*-Numbers. *Turkish Journal of Science*, *5* (2), 147-156.

Erdağ, Ö. & Deveci, Ö. (2020b). On The Connections Between Padovan Numbers and Padovan *p*-Numbers. *International Journal of Open Problems in Computer Science and Mathematics*, *13* (4), 33-47.

Erdağ, Ö. & Deveci, Ö. (2021). The Representation and Finite Sums of the Padovan-*p* Jacobsthal Numbers. *Jordan Journal of Mathematics and Statistics*, *15* (3), 507-521.

Erdağ, Ö. & Deveci, Ö. (2022). On the connections between Padovan numbers and Fibonacci *p*-numbers. *Turkish Journal of Science*, *5* (3), 134-141.

Erdağ, Ö., Deveci, Ö. & Shannon A. G. (2020). Matrix Manipulations for Properties of Pell *p*-Numbers and their Generalizations. *Analele Stiintifice ale Universitatii Ovidius Constanta-Seria Matematica*, *28* (3), 89-102.

Horadam, A. F. (1994). Applications of modified Pell numbers to representations. *Ulam Quarterly*, *3* (1), 34-53.

Kilic, E. (2008). The Binet fomula, sums and representations of generalized Fibonacci *p*-numbers. *European Journal of Combinatorics*, *29* (3), 701-711.

Kılıc, E. (2009). The generalized Pell *(p,i)*-numbers and their Binet formulas, combinatorial representations, sums. *Chaos, Solitons Fractals*, *40*(4), 2047-2063.

Kilic, E. & Tasci, D. (2006). The generalized Binet formula, representation and sums of the generalized order-*k* Pell numbers. *Taiwanese Journal of Mathematics*, *10*(6), 1661-1670.

Lancaster, P. & Tismenetsky, M. (1985). *The theory of matrices: with applications.* Elsevier.

Lidl, R. & Niederreiter, H. (1986). *Introduction to finite fields and their applications.* Cambridge UP.

Shannon, A. G., Erdağ, Ö. & Deveci, Ö. (2021). On the connections between Pell numbers and Fibonacci *p*-numbers. *Notes on Number Theory and Discrete Mathematics*, *27* (1), 148-160.

Shannon, A. G., Horadam, A. F. & Anderson, P. G. (2006). The Auxiliary Equation Associated with the Plastic Number. *Notes on Number Theory and Discrete Mathematics*, *12* (1), 1-12.

Stakhov, A. P. (1999). A Generalization of the Fibonacci *Q*-matrix. *Reports of the National Academy of Sciences of Ukraine*, *9*, 46-49.

Taşçı, D. & Firengiz, M. C. (2010). Incomplete Fibonacci and Lucas *p*-Numbers. *Mathematical and Computer Modelling*, *52* (9-10), 1763-1770.

Tuğlu, N., Koçer, E. G. & Stakhov, A. (2011). Bivariate Fibonacci like *p*-polinomials. *Applied Mathematics and Computation*, *217* (24), 10239-10246.

CHAPTER XI

Parametrıc Versıon Of Modıfıed Bernsteın Operators

Emine GÜVEN

The approximation theory developed under the leadership of Karl Weierstrass is an important step towards a deep understanding of the analysis of functions in mathematics. Weierstrass developed this theory by addressing the problems that arose regarding differentiable and continuous functions in the mid-19th century.

Approximation theory can be broadly defined as the field that studies how closely a function can be imitated by another sequence or series of functions. Addressing some particularly challenging situations in mathematics, Weierstrass showed that a function could be expressed as an infinite sum with a series of other functions. This means that any function can be emulated to the desired precision, despite the complexity of previously determined functions.

Approximation theory was developed because it is an important tool in the mathematical world in efforts to understand the complexity of a function and reach more general conclusions.

Russian mathematician Sergei Natanovich Korovkin, who worked in this field, made important contributions to this field. This theorem, known as the Korovkin Theorem, defines the property of function sequences to approach a certain class under certain conditions and opens the door to many important results in this field.

One of the mathematicians who made significant contributions to Korovkin's approximation theory is David Emmanuel Bernstein. Bernstein defined a class of polynomials that was particularly effective in the approximation of bounded functions and their derivatives. These polynomials increase the applicability to Korovkin's approximation theory by characterizing the general behavior of functions and providing certain approximation properties. Bernstein's works are considered important steps towards building more firmly the foundations of Korovkin's approximation theory in mathematical analysis. These works have been a source of inspiration for many mathematicians in the fields of function theory and approximation theory.

Parametric generalizations of operators within approximation theory is a field that studies the parametric approximation of one class of functions by another class of functions. These generalizations attempt to understand the approximation properties of a function in a broader context, usually by examining parametric families of operators. These studies aim to develop new methods for better approximation of functions in mathematical analysis and applied mathematics.

Parametric generalizations of Bernstein operators are given in [Chen et al., 2017]. Parametric generalizations of operators [Aral et al., 2019], [Cai et al., 2018], [Cai et al., 2021], [Cai et al., 2022], [Çekim et al., 2022], [Kadak et al., 2021], [Kajla et al., 2020], [Mohiuddine et al., 2021], [Mohiuddine et al., 2020], [Özger, 2019], [Srivastava et al., 2019] and [Srivastava et al., 2021].

In this study, the proof of the Korovkin type theorem will be given by defining the parametric modified Bernstein operator for the modified Bernstein operator on the symmetric interval, which was defined and studied by Izgi and Cilo in 2012.

Definition 1.1

$t \in [-1,1]$ and $f \in C[-1,1]$,

$$\boldsymbol{C}_{\eta}(f;\ t) = \frac{1}{2^{\eta}} \sum_{k=0}^{\eta} \binom{\eta}{\kappa} (1+\ t)^{\kappa} (1-\ t)^{\eta-k} f\left(2\frac{\kappa}{\eta} - 1\right)$$
(1)

is called the operator $\boldsymbol{C}_{\eta}(f;\ t)$.

Definition 1.2

$$\wp_{\eta,\kappa} = \binom{\eta}{\kappa} (1-\ t)^{\eta-\kappa} (1+\ t)^{\kappa}$$

Parametric generalization of the modified Bernstein operator for each $f \in C[-1,1]$, $\eta \in \mathbb{N}$ and $t \in [-1,1]$

$$\mathfrak{C}_{\eta,\gamma}(f;\ t) = \sum_{k=0}^{\eta} \wp_{\eta,\kappa}^{(\gamma)} f\left(2\frac{\kappa}{\eta} - 1\right).$$

Where $\eta \geq 1,\ 0 \leq \varpi \leq 1,\ t \in [-1,1]$ and

$\wp_{1,0}^{\gamma}(t) = 1-\ t,\ \ \wp_{1,1}^{(\gamma)}(t) = 1+\ t$ and

to be

$$\wp_{\eta,\kappa}^{(\gamma)}(t) = \left\{ \frac{(1-\gamma)}{2^{\eta-1}} \binom{\eta-2}{\kappa} (1+\ t) + \frac{(1-\gamma)}{2^{\eta-1}} \binom{\eta-2}{\kappa-2} (1-\ t) \right.$$

$$\left. + \frac{\gamma}{2^{\eta}} \binom{\eta}{\kappa} (1+\ t)(1-\ t) \right\} (1+\ t)^{\kappa-1} (1-\ t)^{\eta-\kappa-1},\ \eta \geq 2$$

and

$$\binom{\eta}{\kappa}=\begin{cases}\dfrac{\eta!}{(\eta-\kappa)!\,\kappa!} & , \quad 0\le\kappa\le\eta\\ 0 & , \quad other\end{cases}$$

is. Here

$$\binom{\eta-2}{-2}=\binom{\eta-2}{-1}=0$$

is.

$$\binom{\eta-2}{\kappa}=\left(1-\tfrac{\kappa}{\eta-1}\right)\binom{\eta-1}{\kappa} \text{ and } \binom{\eta-2}{\kappa-2}=\tfrac{\kappa}{\eta-1}\binom{\eta-1}{\kappa}$$

it is clear that it is.

Lemma 1.1

For every $f\in C[-1,1]$, $\eta\in\mathbb{N}$ and $t\in[-1,1]$

$$g\left(2\frac{\kappa}{\eta}-1\right)=f\left(2\frac{\kappa}{\eta}-1\right)\left(1-\frac{\kappa}{\eta-1}\right)+f\left(2\frac{\kappa+1}{\eta}-1\right)\frac{\kappa}{\eta-1}$$

So,

$$\mathfrak{C}_{\eta,\gamma}(f;\,t)=(1-\gamma)\sum_{\kappa=0}^{\eta-1}g\left(2\frac{\kappa}{\eta}-1\right)(1-\,t)^{\eta-\kappa-1}\binom{\eta-1}{\kappa}(1+\,t)^{\kappa}$$

$$+\gamma\sum_{\kappa=0}^{\eta}f\left(2\frac{\kappa}{\eta}-1\right)(1-\,t)^{\eta-\kappa}\binom{\eta}{\kappa}(1+\,t)^{\kappa}$$

is.

Proof:

$$\mathfrak{C}_{\eta,\gamma}(f;\,t)=\sum_{\kappa=0}^{\eta}\left\{\frac{(1-\gamma)}{2^{\eta-1}}\binom{\eta-2}{\kappa}(1+\,t)+\frac{(1-\gamma)}{2^{\eta-1}}\binom{\eta-2}{\kappa-2}(1-\,t)\right.$$

$$+\frac{\gamma}{2^{\eta}}\binom{\eta}{\kappa}(1+\ t)(1-\ t)\Big\}(1-\ t)^{\eta-\kappa-1}(1+\ t)^{\kappa-1}f\left(2\frac{\kappa}{\eta}-1\right)$$

$$=\sum_{\kappa=0}^{\eta}\frac{(1-\gamma)}{2^{\eta-1}}\binom{\eta-2}{\kappa}(1+\ t)\,(1-\ t)^{\eta-\kappa-1}(1+\ t)^{\kappa-1}f\left(2\frac{\kappa}{\eta}-1\right)$$

$$+\sum_{\kappa=0}^{\eta}\frac{(1-\gamma)}{2^{\eta-1}}\binom{\eta-2}{\kappa-2}(1-t)\,(1-t)^{\eta-\kappa-1}(1+t)^{\kappa-1}f\left(2\frac{\kappa}{\eta}-1\right)$$

$$+\sum_{\kappa=0}^{\eta}\frac{\gamma}{2^{\eta}}\binom{\eta}{\kappa}(1+t)(1-t)(1-t)^{\eta-\kappa-1}\,(1+t)^{\kappa-1}f\left(2\frac{\kappa}{\eta}-1\right)$$

$$=\sum_{\kappa=0}^{\eta}\frac{(1-\gamma)}{2^{\eta-1}}\binom{\eta-2}{\kappa}(1-t)^{\eta-\kappa-1}(1+t)^{\kappa}f\left(2\frac{\kappa}{\eta}-1\right)$$

$$+\sum_{\kappa=0}^{\eta}\frac{(1-\gamma)}{2^{\eta-1}}\binom{\eta-2}{\kappa-2}(1-t)^{\eta-\kappa}(1+t)^{\kappa-1}f\left(2\frac{\kappa}{\eta}-1\right)$$

$$+\sum_{\kappa=0}^{\eta}\frac{\gamma}{2^{\eta}}\binom{\eta}{\kappa}(1-t)^{\eta-k}(1+t)^{\kappa}f\left(2\frac{\kappa}{\eta}-1\right)$$

$$=(1-\gamma)\sum_{\kappa=0}^{\eta}\frac{1}{2^{\eta-1}}(1-t)^{\eta-\kappa-1}\binom{\eta-2}{\kappa}(1+t)^{\kappa}f\left(2\frac{\kappa}{\eta}-1\right)$$

$$+(1-\gamma)\sum_{\kappa=0}^{\eta}\frac{1}{2^{\eta-1}}(1-t)^{\eta-\kappa}\binom{\eta-2}{\kappa-2}(1+t)^{\kappa-1}f\left(2\frac{\kappa}{\eta}-1\right)$$

$$+\gamma\sum_{\kappa=0}^{\eta}\frac{1}{2^{\eta}}(1-t)^{\eta-\kappa}\binom{\eta}{\kappa}(1+t)^{\kappa}f\left(2\frac{\kappa}{\eta}-1\right)$$

$$=(1-\gamma)(c_1+c_2)+\gamma\sum_{\kappa=0}^{\eta}\frac{1}{2^{\eta}}(1-t)^{\eta-\kappa}\binom{\eta}{\kappa}(1+t)^{\kappa}f\left(2\frac{\kappa}{\eta}-1\right)$$

$$c_1 = \sum_{\kappa=0}^{\eta} \frac{1}{2^{\eta-1}} (1-t)^{\eta-\kappa-1} \binom{\eta-2}{\kappa} (1+t)^{\kappa} f\left(2\frac{\kappa}{\eta}-1\right)$$

$$c_2 = \sum_{\kappa=0}^{\eta} \frac{1}{2^{\eta-1}} (1-t)^{\eta-\kappa} \binom{\eta-2}{\kappa-2} (1+t)^{\kappa-1} f\left(2\frac{\kappa}{\eta}-1\right)$$

$c_1$is zero for $\eta = \kappa$, c_2, is zero for $\kappa = 0$. Let's

$$c_1 = \sum_{\kappa=0}^{\eta-1} \frac{1}{2^{\eta-1}} (1-t)^{\eta-\kappa-1} \binom{\eta-2}{\kappa} (1+t)^{\kappa} f\left(2\frac{\kappa}{\eta}-1\right)$$

$$c_2 = \sum_{\kappa=1}^{\eta} \frac{1}{2^{\eta-1}} (1-t)^{\eta-\kappa} \binom{\eta-2}{\kappa-2} (1+t)^{\kappa-1} f\left(2\frac{\kappa}{\eta}-1\right)$$

is getting.

$$c_1 = \sum_{\kappa=0}^{\eta-1} \frac{1}{2^{\eta-1}} (1-t)^{\eta-\kappa-1} \binom{\eta-2}{\kappa} (1+t)^{\kappa} f\left(2\frac{\kappa}{\eta}-1\right)$$

$$= \frac{1}{2^{\eta-1}} \sum_{\kappa=0}^{\eta-1} \left(1-\frac{\kappa}{\eta-1}\right) \binom{\eta-1}{\kappa} (1-t)^{\eta-\kappa-1} (1+t)^{\kappa} f\left(2\frac{\kappa}{\eta}-1\right)$$

and

$$c_2 = \frac{1}{2^{\eta-1}} \sum_{\kappa=1}^{\eta} (1-t)^{\eta-\kappa} \binom{\eta-2}{\kappa-2} (1+t)^{\kappa-1} f\left(2\frac{\kappa}{\eta}-1\right)$$

$$= \frac{1}{2^{\eta-1}} \sum_{\kappa=0}^{\eta-1} (1-t)^{\eta-\kappa-1} \binom{\eta-2}{\kappa-1} (1+t)^{\kappa} f\left(2\frac{\kappa+1}{\eta}-1\right)$$

$$= \frac{1}{2^{\eta-1}} \sum_{\kappa=0}^{\eta-1} \frac{\kappa}{\eta-1} (1-t)^{\eta-\kappa-1} \binom{\eta-1}{\kappa} (1+t)^{\kappa} f\left(2\frac{\kappa+1}{\eta}-1\right)$$

$$c_1 + c_2 = \frac{1}{2^{\eta-1}} \sum_{\kappa=0}^{\eta-1} \left\{ f\left(2\frac{\kappa}{\eta}-1\right)\left(1-\frac{\kappa}{\eta-1}\right) + f\left(2\frac{\kappa+1}{\eta}-1\right)\frac{\kappa}{\eta-1} \right\}$$

$$\times (1-t)^{\eta-\kappa-1} \binom{\eta-1}{\kappa} (1+t)^{\kappa}$$

is found. Here if we write,

$$g\left(2\frac{\kappa}{\eta}-1\right) = f\left(2\frac{\kappa}{\eta}-1\right)\left(1-\frac{\kappa}{\eta-1}\right) + f\left(2\frac{\kappa+1}{\eta}-1\right)\frac{\kappa}{\eta-1}$$

then, it is called modified Bernstein operators parametric version

$$\mathfrak{C}_{\eta,\gamma}(f;\, t) = (1-\gamma) \sum_{\kappa=0}^{\eta-1} \frac{1}{2^{\eta-1}} (1- t)^{\eta-\kappa-1} \binom{\eta-1}{\kappa} (1+ t)^{\kappa} g\left(2\frac{\kappa}{\eta} - 1\right)$$

$$+ \gamma \sum_{\kappa=0}^{\eta} \frac{1}{2^{\eta}} (1- t)^{\eta-k} \binom{\eta}{\kappa} (1+ t)^{\kappa} f\left(2\frac{\kappa}{\eta}-1\right).$$

Lemma 1.2

For $\mathfrak{C}_{\eta,\gamma}(f;\, t)$

$i)\ \mathfrak{C}_{\eta,\gamma}(1;t) = 1,$

$ii)\ \mathfrak{C}_{\eta,\gamma}(t;t) = t,$

$iii)\ \mathfrak{C}_{\eta,\gamma}(t^2;t) = \frac{\eta(\eta-1)+2(\gamma-1)}{\eta^2}t^2 - \frac{\eta+2(1-\gamma)}{\eta^2}$

Proof:

$$i)\ \mathfrak{C}_{\eta,\gamma}(1;t) = (1-\gamma)\sum_{\kappa=0}^{\eta-1}\frac{1}{2^{\eta-1}}(1-t)^{\eta-\kappa-1}\binom{\eta-1}{\kappa}(1+t)^{\kappa}$$

$$+\gamma\sum_{\kappa=0}^{\eta}\frac{1}{2^{\eta}}(1-t)^{\eta-\kappa}\binom{\eta}{\kappa}(1+t)^{\kappa}$$

$$= (1-\gamma)\frac{1}{2^{\eta-1}}2^{\eta-1} + \gamma = 1.$$

$$ii)\ \mathfrak{C}_{\eta,\gamma}(t;t) = (1-\gamma)\sum_{\kappa=0}^{\eta-1}\frac{1}{2^{\eta-1}}(1-t)^{\eta-\kappa-1}\binom{\eta-1}{\kappa}(1+t)^{\kappa}g\left(2\frac{\kappa}{\eta}-1\right)$$

$$+\gamma\sum_{\kappa=0}^{\eta}\frac{1}{2^{\eta}}(1-t)^{\eta-\kappa}\binom{\eta}{k}(1+t)^{\kappa}f\left(2\frac{\kappa}{\eta}-1\right)$$

$$= (1-\gamma)\sum_{\kappa=0}^{\eta-1}\frac{1}{2^{\eta-1}}(1-t)^{\eta-\kappa-1}\binom{\eta-1}{\kappa}(1+t)^{\kappa}$$

$$\times\left[f\left(2\frac{\kappa}{\eta}-1\right)\left(1-\frac{\kappa}{\eta-1}\right)+f\left(2\frac{\kappa+1}{\eta}-1\right)\frac{\kappa}{\eta-1}\right]$$

$$+\gamma\sum_{\kappa=0}^{\eta}\frac{1}{2^{\eta}}(1-t)^{\eta-\kappa}\binom{\eta}{\kappa}(1+t)^{\kappa}f\left(2\frac{\kappa}{\eta}-1\right)$$

$$= (1-\gamma)\sum_{\kappa=0}^{\eta-1}\frac{1}{2^{\eta-1}}(1-t)^{\eta-\kappa-1}\binom{\eta-1}{\kappa}(1+t)^{\kappa}$$

$$\times\left[\left(2\frac{\kappa}{\eta}-1\right)\left(1-\frac{\kappa}{\eta-1}\right)+\left(2\frac{\kappa+1}{\eta}-1\right)\frac{\kappa}{\eta-1}\right]$$

$$+\gamma\sum_{\kappa=0}^{\eta}\frac{1}{2^{\eta}}(1-t)^{\eta-\kappa}\binom{\eta}{\kappa}(1+t)^{\kappa}\left(2\frac{\kappa}{\eta}-1\right)$$

$$=(1-\gamma)\sum_{\kappa=0}^{\eta-1}\frac{1}{2^{\eta-1}}(1-t)^{\eta-\kappa-1}\binom{\eta-1}{\kappa}(1+t)^{\kappa}$$

$$\times\left[\left(\frac{2\kappa-\eta}{\eta}\right)\left(\frac{\eta-1-\kappa}{\eta-1}\right)+\left(\frac{2\kappa+2-\eta}{\eta}\right)\frac{\kappa}{\eta-1}\right]$$

$$+\gamma\sum_{\kappa=0}^{\eta}\frac{1}{2^{\eta}}(1-t)^{\eta-\kappa}\binom{\eta}{\kappa}(1+t)^{\kappa}\left(\frac{2\kappa}{\eta}-1\right)$$

$$=(1-\gamma)\sum_{\kappa=0}^{\eta-1}\frac{1}{2^{\eta-1}}(1-t)^{\eta-\kappa-1}\binom{\eta-1}{\kappa}(1+t)^{\kappa}$$

$$\times\left[\frac{2\kappa\eta-2\kappa-2\kappa^2-\eta^2+\eta+\eta\kappa+2\kappa^2+2\kappa-\eta\kappa}{\eta(\eta-1)}\right]$$

$$+\gamma\frac{2}{2^{\eta}\eta}\sum_{\kappa=0}^{\eta}\kappa(1-t)^{\eta-\kappa}\binom{\eta}{\kappa}(1+t)^{\kappa}$$

$$-\gamma\sum_{\kappa=0}^{\eta}\frac{1}{2^{\eta}}(1-t)^{\eta-\kappa}\binom{\eta}{\kappa}(1+t)^{\kappa}$$

$$=(1-\gamma)\sum_{\kappa=0}^{\eta-1}\frac{1}{2^{\eta-1}}(1-t)^{\eta-\kappa-1}\binom{\eta-1}{\kappa}(1+t)^{\kappa}\left[\frac{2\kappa\eta-\eta^2+\eta}{\eta(\eta-1)}\right]$$

$$+\gamma\frac{2}{2^{\eta}\eta}\sum_{\kappa=0}^{\eta}\kappa(1-t)^{\eta-\kappa}\binom{\eta}{\kappa}(1+t)^{\kappa}$$

$$-\ \gamma\sum_{\kappa=0}^{\eta}\frac{1}{2^{\eta}}(1-t)^{\eta-\kappa}\binom{\eta}{\kappa}(1+t)\kappa^{k}$$

$$=(1-\gamma)\sum_{\kappa=0}^{\eta-1}\frac{1}{2^{\eta-1}}(1-t)^{\eta-\kappa-1}\binom{\eta-1}{\kappa}(1+t)^{\kappa}\left[\frac{\eta(2\kappa-\eta+1)}{\eta(\eta-1)}\right]$$

$$+\gamma\frac{2}{2^{\eta}\eta}\sum_{\kappa=0}^{\eta}\kappa\,(1-t)^{\eta-\kappa}\binom{\eta}{\kappa}(1+t)^{\kappa}-\ \gamma$$

$$=(1-\gamma)\sum_{\kappa=0}^{\eta-1}\frac{1}{2^{\eta-1}}(1-t)^{\eta-\kappa-1}\binom{\eta-1}{\kappa}(1+t)^{\kappa}\left[\frac{2\kappa-\eta+1}{\eta-1}\right]$$

$$+\gamma\frac{2}{2^{\eta}\eta}\sum_{\kappa=0}^{\eta}\kappa(1-t)^{\eta-\kappa}\binom{\eta}{\kappa}(1+t)^{\kappa}-\ \gamma$$

$$=(1-\gamma)\sum_{\kappa=0}^{\eta-1}\frac{1}{2^{\eta-1}}(1-t)^{\eta-\kappa-1}\binom{\eta-1}{\kappa}(1+t)^{\kappa}\left[\frac{2\kappa}{\eta-1}-1\right]$$

$$+\gamma\frac{2}{2^{\eta}\eta}\sum_{\kappa=0}^{\eta}\kappa\,(1-t)^{\eta-\kappa}\frac{\eta!}{(\eta-\kappa)!\,\kappa!}(1+t)^{\kappa}-\ \gamma$$

$$=\frac{2(1-\gamma)}{(\eta-1)2^{\eta-1}}\sum_{\kappa=0}^{\eta-1}\kappa\,(1-t)^{\eta-\kappa-1}\binom{\eta-1}{\kappa}(1+t)^{\kappa}$$

$$-\frac{(1-\gamma)}{2^{\eta-1}}\sum_{\kappa=0}^{\eta-1}(1-t)^{\eta-\kappa-1}\binom{\eta-1}{\kappa}(1+t)^{\kappa}$$

$$+\gamma\frac{2}{2^{\eta}\eta}\sum_{\kappa=1}^{\eta}\kappa\,(1-t)^{\eta-\kappa}\frac{\eta!}{(\eta-\kappa)!\,\kappa!}(1+t)^{\kappa}-\gamma$$

$$=\frac{2(1-\gamma)}{(\eta-1)2^{\eta-1}}\sum_{\kappa=1}^{\eta-1}\kappa\,(1-t)^{\eta-\kappa-1}\frac{(\eta-1)!}{(\eta-\kappa-1)!\,\kappa!}(1+t)^{\kappa}$$

$$-\frac{(1-\gamma)}{2^{\eta-1}}\sum_{\kappa=0}^{\eta-1}(1-t)^{\eta-\kappa-1}\frac{(\eta-1)!}{(\eta-\kappa-1)!\,\kappa!}(1+t)^{\kappa}$$

$$+\gamma\frac{2}{2^{\eta}\eta}\sum_{\kappa=0}^{\eta-1}(1-t)^{\eta-\kappa-1}\frac{\eta!}{(\eta-\kappa-1)!\,\kappa!}(1+t)^{\kappa+1}-\gamma$$

$$=\frac{2(1-\gamma)}{(\eta-1)2^{\eta-1}}\sum_{\kappa=0}^{\eta-2}(1-t)^{\eta-\kappa-2}\frac{(\eta-1)!}{(\eta-\kappa-2)!\,\kappa!}(1+t)^{\kappa+1}$$

$$-\frac{(1-\gamma)}{2^{\eta-1}}\sum_{\kappa=0}^{\eta-1}(1-t)^{\eta-\kappa-1}\frac{(\eta-1)!}{(\eta-\kappa-1)!\,\kappa!}(1+t)^{\kappa}$$

$$+\gamma\frac{2}{2^{\eta}\eta}\sum_{\kappa=0}^{\eta-1}(1-t)^{\eta-\kappa-1}\frac{\eta!}{(\eta-\kappa-1)!\,\kappa!}(1+t)^{\kappa+1}-\gamma$$

$$=\frac{2(1-\gamma)(1+t)(\eta-1)}{(\eta-1)2^{\eta-1}}\sum_{\kappa=0}^{\eta-2}(1-t)^{\eta-\kappa-2}\frac{(\eta-2)!}{(\eta-\kappa-2)!\,\kappa!}(1+t)^{\kappa}$$

$$-\frac{(1-\gamma)}{2^{\eta-1}}\sum_{\kappa=0}^{\eta-1}(1-t)^{\eta-k-1}\frac{(\eta-1)!}{(\eta-\kappa-1)!\,k!}(1+t)^{k}$$

$$+\gamma\frac{2(1+t)\eta}{2^{\eta}\eta}\sum_{k=0}^{\eta-1}(1-t)^{\eta-\kappa-1}\frac{(\eta-1)!}{(\eta-\kappa-1)!\,\kappa!}(1+t)^{\kappa}-\gamma$$

$$= \frac{2(1-\gamma)(1+t)}{2^{\eta-1}} 2^{\eta-2} - \frac{(1-\gamma)}{2^{\eta-1}} 2^{\eta-1} + \gamma \frac{2(1+t)\eta}{2^{\eta}\eta} 2^{\eta-1} - \gamma$$

$$= (1-\gamma)(1+t) - (1-\gamma) + \gamma(1+t) - \gamma$$

$$= (1-\gamma)(1+t-1) + \gamma(1+t-1)$$

$$= (1-\gamma)t + \gamma t$$

$$\mathfrak{C}_{\eta,\gamma}(t;t) = t$$

is obtained.

$iii)$ If $\mathfrak{C}_{\eta,\gamma}(t^2;t)$ is calculated,

$$\mathfrak{C}_{\eta,\gamma}(t^2;t) = (1-\gamma)\sum_{\kappa=0}^{\eta-1} \frac{1}{2^{\eta-1}} (1-t)^{\eta-\kappa-1} \binom{\eta-1}{\kappa} (1+t)^{\kappa} g\left(2\frac{\kappa}{\eta} - 1\right)$$

$$+ \gamma \sum_{\kappa=0}^{\eta} \frac{1}{2^{\eta}} (1-t)^{\eta-\kappa} \binom{\eta}{\kappa} (1+t)^{\kappa} f\left(2\frac{\kappa}{\eta} - 1\right)$$

$$= (1-\gamma)\sum_{\kappa=0}^{\eta-1} \frac{1}{2^{\eta-1}} (1-t)^{\eta-\kappa-1} \binom{\eta-1}{\kappa} (1+t)^{\kappa}$$

$$\times \left[\left(2\frac{\kappa}{\eta} - 1\right)^2 \left(1 - \frac{\kappa}{\eta-1}\right) + \left(2\frac{\kappa+1}{\eta} - 1\right)^2 \frac{\kappa}{\eta-1}\right]$$

$$+ \gamma \sum_{k=0}^{\eta} \frac{1}{2^{\eta}} (1-t)^{\eta-\kappa} \binom{\eta}{\kappa} (1+t)^{\kappa} \left(2\frac{\kappa}{\eta} - 1\right)^2$$

Here,

$$\left[\left(\frac{2\kappa}{\eta}-1\right)^2\left(1-\frac{\kappa}{\eta-1}\right)+\left(\frac{2\kappa+2}{\eta}-1\right)^2\frac{\kappa}{\eta-1}\right]$$

$$=\left(\frac{4\kappa^2}{\eta^2}-\frac{4\kappa}{\eta}+1\right)\left(1-\frac{\kappa}{\eta-1}\right)+\left(\frac{4k^2+8k+4}{\eta^2}-\frac{4k+4}{\eta}+1\right)\frac{k}{\eta-1}$$

$$=\frac{4k^2}{\eta^2}-\frac{4k}{\eta}+1-\left(\frac{4k^2}{\eta^2}-\frac{4k}{\eta}+1\right)\frac{\kappa}{\eta-1}+\frac{4\kappa^2+8\kappa+4}{\eta^2}\frac{\kappa}{\eta-1}$$
$$-\frac{4\kappa+4}{\eta}\frac{\kappa}{\eta-1}+\frac{\kappa}{\eta-1}$$

$$=\frac{4\kappa^2}{\eta^2}-\frac{4\kappa}{\eta}+1$$
$$+\left[-\frac{4\kappa^2}{\eta^2}+\frac{4\kappa}{\eta}-1+\frac{4\kappa^2+8\kappa+4}{\eta^2}-\frac{4\kappa+4}{\eta}\right.$$
$$\left.+1\right]\frac{\kappa}{\eta-1}$$

$$=\frac{4\kappa^2}{\eta^2}-\frac{4\kappa}{\eta}+1$$
$$+\left[-\frac{4\kappa^2}{\eta^2}+\frac{4\kappa}{\eta}-1+\frac{4\kappa^2}{\eta^2}+\frac{8\kappa}{\eta^2}+\frac{4}{\eta^2}-\frac{4\kappa}{\eta}-\frac{4}{\eta}\right.$$
$$\left.+1\right]\frac{\kappa}{\eta-1}$$

$$=\frac{4\kappa^2}{\eta^2}-\frac{4\kappa}{\eta}+1+\left[\frac{8\kappa}{\eta^2}+\frac{4}{\eta^2}-\frac{4}{\eta}\right]\frac{\kappa}{\eta-1}$$

$$=\frac{4\kappa^2(\eta-1)}{\eta^2(\eta-1)}-\frac{4\kappa\eta(\eta-1)}{\eta^2(\eta-1)}+\frac{8\kappa^2}{\eta^2(\eta-1)}+\frac{4\kappa}{\eta^2(\eta-1)}-\frac{4\kappa\eta}{\eta^2(\eta-1)}+1$$

$$=\frac{4\kappa^2\eta-4\kappa^2-4\kappa\eta^2+4\kappa\eta+8\kappa^2+4\kappa-4\kappa\eta}{\eta^2(\eta-1)}+1$$

$$=\frac{4\kappa^2\eta+4k\kappa^2-4\kappa\eta^2+4\kappa}{\eta^2(\eta-1)}+1$$

$$= \frac{4\kappa^2(\eta+1) - 4\kappa(\eta^2-1)}{\eta^2(\eta-1)} + 1$$

$$= \frac{4\kappa^2(\eta+1)}{\eta^2(\eta-1)} - \frac{4\kappa(\eta-1)(\eta+1)}{\eta^2(\eta-1)} + 1$$

$$= \frac{4\kappa^2(\eta+1)}{\eta^2(\eta-1)} - \frac{4\kappa(\eta+1)}{\eta^2} + 1$$

if we substitue in $\mathfrak{C}_{n,\gamma}(t^2;t)$

$\mathfrak{C}_{\eta,\gamma}(t^2;t) =$

$$= (1-\gamma)\sum_{\kappa=0}^{\eta-1} \frac{1}{2^{\eta-1}}(1-t)^{\eta-\kappa-1}\binom{\eta-1}{\kappa}(1+t)^{\kappa}\left[\frac{4(\eta+1)}{\eta^2(\eta-1)}\kappa^2 - \frac{4(\eta+1)}{\eta^2}\kappa + 1\right]$$

$$+ \gamma\sum_{\kappa=0}^{\eta} \frac{1}{\eta}(1-t)^{\eta-\kappa}\binom{\eta}{\kappa}(1+t)^{k}\left(\frac{4\kappa^2}{\eta^2} - \frac{4\kappa}{\eta} + 1\right)$$

$$= (1-\gamma)\sum_{\kappa=0}^{\eta-1} \frac{1}{2^{\eta-1}}(1-t)^{\eta-\kappa-1}\binom{\eta-1}{\kappa}(1+t)^{\kappa}\frac{4(\eta+1)}{\eta^2(\eta-1)}\kappa^2$$

$$-(1-\gamma)\sum_{\kappa=0}^{\eta-1} \frac{1}{2^{\eta-1}}(1-t)^{\eta-\kappa-1}\binom{\eta-1}{\kappa}(1+t)^{\kappa}\frac{4(\eta+1)}{\eta^2}\kappa$$

$$+(1-\gamma)\sum_{\kappa=0}^{\eta-1} \frac{1}{2^{\eta-1}}(1-t)^{\eta-\kappa-1}\binom{\eta-1}{\kappa}(1+t)^{\kappa}$$

$$+ \gamma\sum_{\kappa=0}^{\eta} \frac{1}{2^{\eta}}(1-t)^{\eta-\kappa}\binom{\eta}{\kappa}(1+t)^{\kappa}\frac{4\kappa^2}{\eta^2}$$

$$- \gamma \sum_{\kappa=0}^{\eta} \frac{1}{2^{\eta}} (1-t)^{\eta-\kappa} \binom{\eta}{\kappa} (1+t)^{\kappa} \frac{4\kappa}{\eta}$$

$$+ \ \gamma \sum_{\kappa=0}^{\eta} \frac{1}{2^{\eta}} (1-t)^{\eta-\kappa} \binom{\eta}{\kappa} (1+t)^{\kappa}$$

$$= \frac{4(\eta+1)(1-\gamma)}{2^{\eta-1}\eta^{2}(\eta-1)} \sum_{\kappa=0}^{\eta-1} \kappa^{2} (1-t)^{\eta-\kappa-1} \binom{\eta-1}{\kappa} (1+t)^{\kappa}$$

$$- \frac{4(1-\gamma)(\eta+1)}{2^{\eta-1}\eta^{2}} \sum_{\kappa=0}^{\eta-1} \kappa (1-t)\eta^{n-\kappa-1} \binom{\eta-1}{\kappa} (1+t)^{\kappa}$$

$$+ \frac{(1-\gamma)}{2^{\eta-1}} \sum_{\kappa=0}^{\eta-1} (1-t)^{\eta-\kappa-1} \binom{n-1}{\kappa} (1+t)^{\kappa}$$

$$+ \ \frac{4\gamma}{\eta^{2}2^{\eta}} \sum_{\kappa=0}^{\eta} \kappa^{2} (1-t)^{\eta-\kappa} \binom{\eta}{\kappa} (1+t)^{\kappa}$$

$$- \frac{4\gamma}{\eta 2^{\eta}} \sum_{\kappa=0}^{\eta} \kappa (1-t)^{\eta-\kappa} \binom{\eta}{\kappa} (1+t)^{\kappa} + \frac{\gamma}{2^{\eta}} 2^{\eta}$$

$$= \frac{(\eta+1)(1-\gamma)}{2^{n-3}\eta^{2}(\eta-1)} \sum_{\kappa=0}^{\eta-1} \kappa^{2} (1-t)^{\eta-\kappa-1} \binom{\eta-1}{\kappa} (1+t)^{\kappa}$$

$$+ \frac{\gamma}{\eta^{2}2^{\eta-2}} \sum_{\kappa=0}^{\eta} \kappa^{2} (1-t)^{\eta-\kappa} \binom{\eta}{\kappa} (1+t)^{\kappa}$$

$$- \frac{(1-\gamma)(\eta+1)}{2^{\eta-3}\eta^{2}} \sum_{\kappa=0}^{\eta-1} \kappa(1-t)^{\eta-\kappa-1} \binom{\eta-1}{\kappa} (1+t)^{\kappa}$$

$$- \frac{\gamma}{\eta 2^{\eta-2}} \sum_{\kappa=0}^{\eta} \kappa (1-t)^{n-\kappa} \binom{\eta}{\kappa} (1+t)^{\kappa} + \frac{(1-\gamma)}{2^{\eta-1}} 2^{\eta-1} + \ \gamma$$

$$= \frac{(\eta+1)(1-\gamma)}{2^{\eta-3}\eta^2(\eta-1)} \sum_{\kappa=2}^{\eta-1} \kappa(\kappa-1)\,(1-t)^{\eta-\kappa-1} \frac{(\eta-1)!}{(\eta-\kappa-1)!\,\kappa!}(1+t)^{\kappa}$$

$$+ \frac{(\eta+1)(1-\gamma)}{2^{n-3}\eta^2(\eta-1)} \sum_{\kappa=1}^{\eta-1} \kappa\,(1-t)^{\eta-\kappa-1} \frac{(\eta-1)!}{(\eta-\kappa-1)!\,\kappa!}(1+t)^{\kappa}$$

$$+ \frac{\gamma}{\eta^2 2^{\eta-2}} \sum_{\kappa=2}^{\eta} \kappa(\kappa-1)(1-t)^{\eta-\kappa} \frac{\eta!}{(\eta-\kappa)!\,\kappa!}(1+t)^{\kappa}$$

$$+ \frac{\gamma}{\eta^2 2^{\eta-2}} \sum_{\kappa=1}^{\eta} \kappa\,(1-t)^{\eta-\kappa} \frac{\eta!}{(\eta-\kappa)!\,\kappa!}(1+t)^{\kappa}$$

$$- \frac{(1-\gamma)(\eta+1)}{2^{\eta-3}\eta^2} \sum_{\kappa=1}^{\eta-1} \kappa(1-t)^{\eta-\kappa-1} \frac{(\eta-1)!}{(\eta-\kappa-1)!\,\kappa!}(1+t)^{\kappa}$$

$$- \frac{\gamma}{\eta 2^{\eta-2}} \sum_{\kappa=1}^{\eta} \kappa\,(1-t)^{\eta-\kappa} \frac{\eta!}{(\eta-\kappa)!\,\kappa!}(1+t)^{\kappa} + 1$$

$$= \frac{(\eta+1)(1-\gamma)}{2^{\eta-3}\eta^2(\eta-1)} \sum_{\kappa=2}^{\eta-1} (1-t)^{\eta-\kappa-1} \frac{(\eta-1)!}{(\eta-\kappa-1)!\,(\kappa-2)!}(1+t)^{\kappa}$$

$$+ \frac{(\eta+1)(1-\gamma)}{2^{\eta-3}\eta^2(\eta-1)} \sum_{\kappa=1}^{\eta-1} (1-t)^{\eta-\kappa-1} \frac{(\eta-1)!}{(\eta-\kappa-1)!\,(\kappa-1)!}(1+t)^{\kappa}$$

$$+ \frac{\gamma}{\eta^2 2^{\eta-2}} \sum_{\kappa=2}^{\eta} (1-t)^{\eta-\kappa} \frac{\eta!}{(\eta-\kappa)!\,(\kappa-2)!}(1+t)^{\kappa}$$

$$+ \frac{\gamma}{\eta^2 2^{\eta-2}} \sum_{\kappa=1}^{\eta} (1-t)^{\eta-\kappa} \frac{\eta!}{(\eta-\kappa)!\,(\kappa-1)!}(1+t)^{\kappa}$$

$$-\frac{(1-\gamma)(\eta+1)}{2^{\eta-3}\eta^2}\sum_{\kappa=1}^{\eta-1}(1-t)^{\eta-\kappa-1}\frac{(\eta-1)!}{(\eta-\kappa-1)!\,(\kappa-1)!}(1+t)^{\kappa}$$

$$-\frac{\gamma}{\eta 2^{\eta-2}}\sum_{\kappa=1}^{\eta}(1-t)^{\eta-\kappa}\frac{\eta!}{(\eta-\kappa)!\,(\kappa-1)!}(1+t)^{\kappa}+1$$

$$=\frac{(\eta+1)(1-\gamma)}{2^{\eta-3}\eta^2(\eta-1)}\sum_{\kappa=0}^{\eta-3}(1-t)^{\eta-\kappa-3}\frac{(\eta-1)!}{(\eta-\kappa-3)!\,\kappa!}(1+t)^{\kappa+2}$$

$$+\frac{(\eta+1)(1-\gamma)}{2^{\eta-3}\eta^2(\eta-1)}\sum_{\kappa=0}^{\eta-2}(1-t)^{\eta-\kappa-2}\frac{(\eta-1)!}{(\eta-\kappa-2)!\,\kappa!}(1+t)^{\kappa+1}$$

$$+\frac{\gamma}{\eta^2 2^{\eta-2}}\sum_{\kappa=0}^{\eta-2}(1-t)^{\eta-\kappa-2}\frac{\eta!}{(\eta-\kappa-2)!\,\kappa!}(1+t)^{\kappa+2}$$

$$+\frac{\gamma}{\eta^2 2^{\eta-2}}\sum_{\kappa=0}^{\eta-1}(1-t)^{\eta-\kappa-1}\frac{\eta!}{(\eta-\kappa-1)!\,\kappa!}(1+t)^{\kappa+1}$$

$$-\frac{(1-\gamma)(\eta+1)}{2^{\eta-3}\eta^2}\sum_{\kappa=0}^{\eta-2}(1-t)^{\eta-\kappa-2}\frac{(\eta-1)!}{(\eta-\kappa-2)!\,\kappa!}(1+t)^{\kappa+1}$$

$$-\frac{\gamma}{\eta 2^{\eta-2}}\sum_{\kappa=0}^{\eta-1}(1-t)^{\eta-\kappa-1}\frac{\eta!}{(\eta-\kappa-1)!\,\kappa!}(1+t)^{\kappa+1}+1$$

$$=\frac{(\eta+1)(1-\gamma)(\eta-1)(\eta-2)(1+t)^2}{2^{\eta-3}\eta^2(\eta-1)}\sum_{\kappa=0}^{\eta-3}(1-t)^{\eta-\kappa-3}\frac{(\eta-3)!}{(\eta-\kappa-3)!\,\kappa!}(1+t)^{\kappa}$$

$$+\frac{(\eta+1)(1-\gamma)(\eta-1)(1+t)}{2^{\eta-3}\eta^2(\eta-1)}\sum_{\kappa=0}^{\eta-2}(1-t)^{\eta-\kappa-2}\frac{(\eta-2)!}{(\eta-\kappa-2)!\,\kappa!}(1+t)^{\kappa}$$

$$+\frac{\gamma\eta(\eta-1)(1+t)^2}{\eta^2 2^{\eta-2}}\sum_{\kappa=0}^{\eta-2}(1-t)^{\eta-\kappa-2}\frac{(\eta-2)!}{(\eta-\kappa-2)!\,\kappa!}(1+t)^{\kappa}$$

$$+\frac{\gamma\eta(1+t)}{\eta^2 2^{\eta-2}}\sum_{\kappa=0}^{\eta-1}(1-t)^{\eta-\kappa-1}\frac{(\eta-1)!}{(\eta-\kappa-1)!\,\kappa!}(1+t)^{\kappa}$$

$$-\frac{(1-\gamma)(\eta+1)(\eta-1)(1+t)}{2^{\eta-3}\eta^2}\sum_{\kappa=0}^{\eta-2}(1-t)^{\eta-\kappa-2}\frac{(\eta-2)!}{(\eta-\kappa-2)!\,k!}(1+t)^{\kappa}$$

$$-\frac{\gamma\eta(1+t)}{\eta 2^{\eta-2}}\sum_{\kappa=0}^{\eta-1}(1-t)^{\eta-\kappa-1}\frac{(\eta-1)!}{(\eta-\kappa-1)!\,\kappa!}(1+t)^{\kappa}+1$$

$$=\frac{(1-\gamma)(\eta+1)(\eta-2)(1+t)^2}{2^{\eta-3}\eta^2}2^{\eta-3}+\frac{(\eta+1)(1-\gamma)(1+t)}{2^{\eta-3}\eta^2}2^{\eta-2}$$

$$+\frac{\gamma(\eta-1)(1+t)^2}{\eta 2^{\eta-2}}2^{\eta-2}+\frac{\gamma(1+t)}{\eta 2^{n-2}}2^{n-1}-\frac{(1-\gamma)(\eta+1)(\eta-1)(1+t)}{2^{\eta-3}\eta^2}2^{\eta-2}$$

$$-\frac{\gamma(1+t)}{2^{\eta-2}}2^{\eta-1}+1$$

$$=\frac{(1-\gamma)(\eta+1)(\eta-2)(1+t)^2}{\eta^2}+\frac{2(\eta+1)(1-\gamma)(1+t)}{\eta^2}$$

$$+\frac{\gamma(\eta-1)(1+t)^2}{\eta}+\frac{2\gamma(1+t)}{\eta}-\frac{2(1-\gamma)(\eta+1)(\eta-1)(1+t)}{\eta^2}$$

$$-2\gamma(1+t)+1$$

$$= \frac{(1-\gamma)(\eta+1)(\eta-2)(1+t)^2 + 2(\eta+1)(1-\gamma)(1+t)}{\eta^2}$$

$$- \frac{2(1-\gamma)(\eta+1)(\eta-1)(1+t)}{\eta^2}$$

$$+ \frac{\gamma(\eta-1)(1+t)^2 + 2\gamma(1+t)}{\eta} - 2\gamma(1+t) + 1$$

$$= \frac{(1-\gamma)(\eta+1)(\eta-2)(1+t)^2 + [1-(\eta-1)]2(\eta+1)(1-\gamma)(1+t)}{\eta^2}$$

$$+ \frac{\gamma(\eta-1)\eta(1+t)^2 + 2\gamma\eta(1+t) - 2\gamma(1+t)\eta^2 + \eta^2}{\eta^2}$$

$$= \frac{(1-\gamma)(\eta+1)(\eta-2)(1+t)^2 + (2-\eta)2(\eta+1)(1-\gamma)(1+t)}{\eta^2}$$

$$+ \frac{\gamma(\eta-1)\eta(1+t)^2 + 2\gamma\eta(1+t) - 2\gamma(1+t)\eta^2 + \eta^2}{\eta^2}$$

$$= \frac{(1-\gamma)(\eta+1)(\eta-2)[(1+t)^2 - 2(1+t)]}{\eta^2}$$

$$+ \frac{\gamma\eta^2(1+t)^2 - \gamma\eta(1+t)^2 + 2\gamma\eta(1+t) - 2\gamma(1+t)\eta^2 + \eta^2}{\eta^2}$$

$$= \frac{(1-\gamma)(\eta+1)(\eta-2)[t^2 + 2t + 1 - 2 - 2t]}{\eta^2}$$

$$+ \frac{\gamma\eta^2(1+t)^2 - \gamma\eta(1+t)^2 + 2\gamma\eta(1+t) - 2\gamma(1+t)\eta^2 + \eta^2}{\eta^2}$$

$$= \frac{(1-\gamma)(\eta+1)(\eta-2)(t^2-1)}{\eta^2}$$

$$+\frac{\gamma\eta^2(1+t)^2-\gamma\eta(1+t)^2+2\gamma\eta(1+t)-2\gamma(1+t)\eta^2+\eta^2}{\eta^2}$$

$$=\frac{(1-\gamma)(\eta+1)(\eta-2)t^2}{\eta^2}-\frac{(1-\gamma)(\eta+1)(\eta-2)}{\eta^2}$$

$$+\frac{\gamma\eta^2(1+2t+t^2)-\gamma\eta(1+2t+t^2)+2\gamma\eta(1+t)-2\gamma(1+t)\eta^2+\eta^2}{\eta^2}$$

$$=\frac{(1-\gamma)(\eta+1)(\eta-2)t^2}{\eta^2}-\frac{(1-\gamma)(\eta+1)(\eta-2)}{\eta^2}$$

$$+\frac{\gamma\eta^2+2\gamma\eta^2t+\gamma\eta^2t^2-\gamma\eta-2\gamma\eta t-\gamma\eta t^2+2\gamma\eta+2\gamma\eta t-2\gamma\eta^2-2\gamma\eta^2t+\eta}{n\eta^2}$$

$$=\frac{(1-\gamma)(\eta+1)(\eta-2)t^2}{\eta^2}-\frac{(1-\gamma)(\eta+1)(\eta-2)}{\eta^2}$$

$$+\frac{\gamma\eta^2t^2-\gamma\eta t^2+\gamma\eta-\gamma\eta^2+\eta^2}{\eta^2}$$

$$=\frac{(1-\gamma)(\eta+1)(\eta-2)t^2+\gamma\eta^2t^2-\gamma\eta t^2}{\eta^2}-\frac{(1-\gamma)(\eta+1)(\eta-2)-\gamma\eta+\gamma\eta^2-\eta^2}{\eta^2}$$

$$=\frac{[(1-\gamma)(\eta^2-\eta-2)+\gamma\eta^2-\gamma\eta]t^2}{\eta^2}-\frac{(1-\gamma)(\eta^2-\eta-2)-\gamma\eta+\gamma\eta^2-\eta^2}{\eta^2}$$

$$=\frac{[n^2-n-2-n^2\gamma+n\gamma+2\gamma+\gamma n^2-\gamma n]t^2}{\eta^2}$$

$$+\frac{\eta^2-\eta-2-\eta^2\gamma+\eta\gamma+2\gamma-\gamma\eta+\gamma\eta^2-\eta^2}{\eta^2}$$

$$=\frac{[\eta^2-\eta-2+2\gamma]\mathcal{t}^2}{\eta^2}-\frac{\eta+2-2\gamma}{\eta^2}$$

$$C_{n,\gamma}(t^2;\mathcal{t})=\frac{\eta(\eta-1)+2(\gamma-1)}{\eta^2}\mathcal{t}^2-\frac{\eta+2(1-\gamma)}{\eta^2}$$

is obtained.

Lemma 1.3

The central moments of the operator $\mathfrak{C}_{n,\gamma}(f;\mathcal{t})$ are as follows.

$i)\mathfrak{C}_{\eta,\gamma}((t-\mathcal{t})^0,x)=1$

$ii)\ \mathfrak{C}_{\eta,\gamma}(t-\mathcal{t},\mathcal{t})=0$

$$iii)\ \mathfrak{C}_{\eta,\gamma}((t-\mathcal{t})^2,\mathcal{t})=\left[\frac{-\eta-2+2\gamma}{\eta^2}\right]x^2-\frac{\eta+2-2\gamma}{\eta^2}$$

Proof:

$i)\mathfrak{C}_{\eta,\gamma}((t-\mathcal{t})^0,\mathcal{t})=\mathfrak{C}_{n,\gamma}(1,\mathcal{t})=1$

$ii)\ \mathfrak{C}_{\eta,\gamma}(t-\mathcal{t},\mathcal{t})=\mathcal{t}-\mathcal{t}=0$

$$iii)\ \mathfrak{C}_{\eta,\gamma}((t-\mathcal{t})^2,\mathcal{t})=\mathfrak{C}_{\eta,\gamma}(t^2,\mathcal{t})-\ 2\mathcal{t}\mathfrak{C}_{\eta,\gamma}(t,\mathcal{t})+\mathcal{t}^2\ \mathfrak{C}_{\eta,\gamma}(1,\mathcal{t})$$

$$=\frac{\eta(\eta-1)+2(\gamma-1)}{\eta^2}\mathcal{t}^2-\frac{\eta+2(1-\gamma)}{\eta^2}-2\mathcal{t}\mathcal{t}+\mathcal{t}^2$$

$$=\frac{\eta(\eta-1)+2(\gamma-1)}{\eta^2}\mathcal{t}^2-\frac{\eta+2(1-\gamma)}{\eta^2}-\ \mathcal{t}^2$$

$$=\left[\frac{\eta(\eta-1)+2(\gamma-1)}{\eta^2}-1\right]\mathcal{t}^2-\frac{\eta+2(1-\gamma)}{\eta^2}$$

$$\mathfrak{C}_{n,\gamma}((t-\mathcal{t})^2,\mathcal{t})=\left[\frac{-\eta+2(\gamma-1)}{\eta^2}\right]\mathcal{t}^2-\frac{\eta+2(1-\gamma)}{\eta^2}$$

is having.

Theorem 1.1

$f \in \mathfrak{C}[-1,1]$, $n \in \mathbb{N}$ and $t \in [-1,1]$. In this case,

$$\lim_{\eta\to\infty} \left\|\mathfrak{C}_{\eta,\gamma}(f;t) - f(t)\right\|_{C[-1,1]} = 0$$

Proof:

For $f(t) = 1$ and $f(t) = t$ the proof is clearly. Now, for $f(t) = t^2$,

$$\lim_{\eta\to\infty} \left\|\mathfrak{C}_{\eta,\gamma}(f;t) - t^2 \right\|_{C[-1,1]} =$$

$$\lim_{\eta\to\infty} \max_{x\in[-1,1]} \left|\mathfrak{C}_{\eta,\gamma}(t^2;t) - t^2\right|$$

$$= \lim_{\eta\to\infty} \max_{x\in[-1,1]} \left|\frac{\eta(\eta-1)+2(\gamma-1)}{\eta^2} t^2 - \frac{\eta+2(1-\gamma)}{\eta^2} - t^2\right|$$

$$= \lim_{\eta\to\infty} \max_{x\in[-1,1]} \left|\frac{\eta(\eta-1)+2(\gamma-1)}{\eta^2} t^2 - \frac{\eta+2(1-\gamma)}{\eta^2} - t^2\right|$$

$$= \lim_{\eta\to\infty} \max_{x\in[-1,1]} \left|\left[\frac{\eta(\eta-1)+2(\gamma-1)}{\eta^2} - 1\right] t^2 - \frac{\eta+2(1-\gamma)}{n^2}\right|$$

$$= \lim_{\eta\to\infty} \max_{x\in[-1,1]} \left|\left[\frac{-\eta+2(\gamma-1)}{\eta^2}\right]\right| t^2 + \lim_{\eta\to\infty} \max_{x\in[-1,1]} \left|\frac{\eta+2(1-\gamma)}{\eta^2}\right|$$

$$= 0$$

is getting. So, we have next form

$$\lim_{\eta\to\infty} \left\|\mathfrak{C}_{\eta,\gamma}(f;t) - f(t)\right\|_{C[-1,1]} = 0.$$

Conclusion

Firstly, important equations that will be used to examine the operator's approximation results have been obtained. Then, the central moment of the operator was calculated. Finally, with the help of Korovkin type theorem, it was shown that every function in space can be approached with the defined operator. In fact, many different features of the defined operator can be examined. For example convexity, monotonicity of operators. In fact, many different features of the defined operator can be examined. for example convexity, monotonicity of operators. Using modulus of continuity we could be calculated of rate of convergence and Vornovskaya type theorem. On the other hand, with this operator, some special sequences and polynomials can be selected and the approximation can be shown visually and numerically.See(Yılmaz & Soykan, 2023), (Aktas, & Soykan, 2023). Also, we On the other hand, this method we use can be applied to different operators in the literature. For example: (Bilgin and Eren 2023), (Bozma and Bilgin, 2023).

References

Aktas, S., & Soykan, Y. (2023). *A Study on the Norms of Toeplitz Matrices with the Generalized Mersenne Numbers.* Archives of Current Research International, 23(7), 143-157.

Aral, A., Erbay, H. (2019). *Parametric generalization of Baskakov operators.* Math Commun. 2019;24:119–131.

Bernstein, S. (1912). *Démo istration du th'eorème de Weierstrass fondée sur le calcul des probabilités.* Сообщенія Харьковскаго математическаго общества, 13(1), 1-2.

Bilgin, N. G., & Eren M.(2023). *Results on bivariate modified (p, q)-Bernstein type operators.* Gazi University Journal of Science, 1-1.

Bozma, G., & Bilgin, NG. *A New Type Bivariate Bernstein Schurer Operators.* Proceeding 6 th International Conference on Innovative Studies of Contemporary Sciences August 1-2, 2022 / Tokyo, Japan

Cai, Q.-B., Lian, B.-Y ., Zhou, G. (2018). *Approximation properties of λ-Bernstein operators.* J. Inequal. Appl. 2018, 61.

Cai, Q. B., & Aslan, R. (2021). *On a new construction of generalized q-Bernstein polynomials based on shape parameter λ.* Symmetry, 13(10), 1919.

Cai, Q.-B., Aslan, R. (2022). *Note on a New Construction of Kantorovich Form $q-$Bernstein Operators Related to Shape Parameter λ.* Computer Modeling in Engineering Sciences. Henderson Vol. 130, Iss. 3, 1479-1493.

Cekim, B., Aktaş, R., Taşdelen, F. (2022). *A Dunkl-Gamma type operator in terms of generalization of two-variable Hermite polynomials.* Indian J Pure Appl Math, 53, 727–735.

Chen, X., Tan, J., Liu, Z., & Xie, J. (2017). *Approximation of functions by a new family of generalized Bernstein* *operators*. Journal of Mathematical Analysis and Applications, 450(1), 244-261.

Cilo, A. Ural, A, Izgi, A, (2012). *Comparison of bernstein polynomials and some modifications* (See page:93), September 2012, Conference: XXV. National Mathematics Symposium - Niğde University (5-8 September (September)

Cilo, A. (Advisor: IZGI, A) (2012). *In [-1,1] ranges Bernstein polynomials approach properties and approach spee*d, Master's Thesis, Harran University, Şanlıurfa.

Hacısalihoğlu, H.H., Hacıyev, A. (1995). *Lineer Pozitif Operatör Dizilerinin Yakınsaklığı,* A.U.F.F. Döner Sermayesi İşletmesi Yayınlar, Ankara, 1-100

Kadak, U., Ozger, F. (2021). *A numerical comparative study of generalized Bernstein- Kantorovich operators.* Mathematical Foundations of Computing. 2021, 4(4): 311- 332. doi: 10.3934/mfc.2021021

Kajla, A., Mursaleen, M., Acar, T.(2020). *Durrmeyer-Type Generalization of Parametric Bernstein Operators.* Symmetry. 2020; 12(7):1141.

Kanat, K., Sofyalıoğlu, M., Erdal, S. *Parametric Generalization of the Modified Bernstein- Kantorovich Operators.* (In Press)

Korovkin, P.P. (1953). *On convergence of linear operators in the space of continuous functions* (Russian). Dokl Akad Nauk SSSR (N.S.) 90:961-–964 .

Mohiuddine, S.A., Ahmad, N., Ozger, F., Alotaibi, A., Hazarika, B. (2021). *Approximation by the Parametric Generalization of Baskakov–Kantorovich Operators Linking with Stancu Operators.* Iran J Sci Technol Trans Sci 45, 593–605

Mohiuddine, S.A., Özger, F. (2020). *Approximation of functions by Stancu variant of Bernstein–Kantorovich operators based on shape parameter α.* Revista de la Real Academia de

Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, 114(2), 70.

Ozger, F. (2019). *Weighted statistical approximation properties of univariate and bivariate λ- Kantorovich operators.* Filomat, 33(11) 3473–3486.

Srivastava, H.M., Ozger, F., Mohiuddine, S.A. (2019). *Construction of Stancu-type Bernstein operators based on Bézier bases with shape parameter λ.* Symmetry 11(3), Article 316.

Srivastava, H. M., Ansari, K.J., Ozger, F., Odemiş Ozger, Z. A. (2021). *Link between Approximation Theory and Summability Methods via Four-Dimensional Infinite Matrices.* Mathematics 2021, 9, 1895.

Yılmaz, B., & Soykan, Y. (2023). *Gaussian Generalized Guglielmo Numbers.* Asian Journal of Advanced Research and Reports, 17(12), 1-18.

Weierstraas, K. (1885). *Uber die analytische Darstellbarkeit sogenannter willkürlicher Functionen einer reellen Veränderlichen.* Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin, 2, 633-639.

CHAPTER XII

Approximation with Gadjiev-Ibragimov Operator on a Mobile Interval

Gurel BOZMA[1]

INTRODUCTION

Studies in the field of approximation theory gained momentum after Weierstrass, who proved the existence of a polynomial converging to every continuous function defined in a finite interval, and Bohman's study investigating the approximation conditions for linear positive operators in the interval [0,1] and the field is largely shaped by showing that in Korovkin's study, it is shown that if the conditions for the test functions are met, the convergence of the operator will be obtained for all functions in space.

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After this important theorem of Korovkin, operators that could be the application of the theorem were constructed and new operators were designed by combining several operators by different researchers and the field was developed rapidly. Here are a few examples of these operators: (Bernstein, 1915), (Dogru, 1997), (Coskun, 2011), (Gonul & Coskun, 2012, 2013), (Kaya & Gonul, 2013), (Deniz & Aral, 2015), (Acar, 2017), (Bilgin & Cetinkaya, 2018), (Gonul Bilgin & Ozgur, 2019), (Bilgin & Eren, 2021), (Bozma & Bars, 2022) (Herdem & Buyukyazici, 2020).

The Gadjiev-Ibragimov operator, which is one of the preferred operators in the studies in the literature, was first defined in (Ibragimov & Gadjiev, 1970) and has been studied from different perspectives by many researchers, especially the students trained by Gadjiev.

In the study, the generalized Gadjiev-Ibragimov operator, which is well known in the literature, is modified to provide suitable properties for test functions over a general interval whose limit is variable-dependent. Important properties of the approximation to continuous functions are introduced with a sequences of linear positive operators defined on a mobile interval. Also, the rate of convergent is calculated numerically by obtaining significant equations for the defined our operators.

The approach speed was calculated with the help of the continuity module by obtaining important equations for the operator. The approximation results obtained with graphical and numerical calculations have been put forward in practice.

METHOD

In this section, the operator to be studied will be defined and important approximation properties will be given.

Definition 2.1 Let ϕ_n and g_n be sequences satisfying the conditions $\lim_{n\to\infty} g_n = \infty$, $\lim_{n\to\infty} \frac{\phi_n}{g_n} = 0$ and $\lim_{n\to\infty} \frac{\phi_n}{g_n+n+\psi} n = 1$, $\lim_{n\to\infty} \frac{n+\mu}{g_n+n+\psi} = 0$.

Let $R_{n,e}(x)$ be a function that satisfies the following conditions depending on the parameters e and n and let $\mu, \psi \in \mathbb{R}^+$.

1-) For all $n \in \mathbb{N}$, $e \in \mathbb{N}_0$ and $\mu < \psi$, also for $x \in \left[0, \frac{n+\mu}{n+\psi}\right]$,

$(-1)^e R_{n,e}(x) \geq 0,$

2-) For every $x \in \left[0, \frac{n+\mu}{n+\psi}\right]$

$\sum_{e=0}^{\infty} R_{n,e}(x) \frac{(-\phi_n)^e}{e!} = 1,$

3-) For every $x \in \left[0, \frac{n+\mu}{n+\psi}\right]$, there is an integer z such that the equality

$R_{n,e}(x) = -nxR_{n+z,e-1}(x),$

will be satisfied and $(n+z) \in \mathbb{N}_0$. In this case, the operator, which is a generalization of the Gadjiev-Ibragimov operator, for every $f \in C\left[0, \frac{n+\mu}{n+\psi}\right]$

$$\tilde{L}_n(f,x) = \sum_{e=0}^{\infty} f\left(\frac{e+n+\mu}{g_n+n+\psi}\right) R_{n,e}(x) \frac{(-\phi_n)^e}{e!}$$

will be in the form. Obviously this operator is linear and positive.

Lemma 2. 1 Let $f \in C\left[0, \frac{n+\mu}{n+\psi}\right]$ The following equations are valid for

$$\tilde{L}_n(f,x) = \sum_{e=0}^{\infty} f\left(\frac{e+n+\mu}{g_n+n+\psi}\right) R_{n,e}(x) \frac{(-\phi_n)^e}{e!}.$$

i) $\tilde{L}_n(1,x) = 1,$

ii)$\tilde{L}_n(h,x) = \frac{n\phi_n}{g_n+n+\psi}x + \frac{n+\mu}{g_n+n+\psi},$

iii) $\tilde{L}_n(h^2,x) = \frac{(\phi_n)^2 n(n+z)}{(g_n+n+\psi)^2}x^2 + \frac{\big((2n+\mu)+1\big)n\phi_n}{(g_n+n+\psi)^2}x$

$+\frac{(n+\mu)^2}{(g_n+n+\psi)^2}.$

Proof According to the 2nd property of the operator, since

$$\sum_{e=0}^{\infty} R_{n,e}(x) \frac{(-\phi_n)^e}{e!} = 1$$

obviously

$$\tilde{L}_n(1,x) = \sum_{e=0}^{\infty} R_{n,e}(x) \frac{(-\phi_n)^e}{e!} = 1$$

is found. Using the 3rd property of the operator; Since $(n+z)\in\mathbb{N}_0$

$$\tilde{L}_n(h,x)=\sum_{e=0}^{\infty}\left(\frac{e+n+\mu}{g_n+n+\psi}\right)R_{n,e}(x)\frac{(-\phi_n)^e}{e!}$$

$$=\sum_{e=1}^{\infty}\left(\frac{e}{g_n+n+\psi}\right)R_{n,e}(x)\frac{(-\phi_n)^e}{e!}$$

$$+\sum_{e=0}^{\infty}\left(\frac{n+\mu}{g_n+n+\psi}\right)R_{n,e}(x)\frac{(-\phi_n)^e}{e!}$$

$$=\sum_{e=1}^{\infty}\left(\frac{1}{g_n+n+\psi}\right)R_{n,e}(x)\frac{(-\phi_n)^e}{(e-1)!}$$

$$+\sum_{e=0}^{\infty}\left(\frac{n+\mu}{g_n+n+\psi}\right)R_{n,e}(x)\frac{(-\phi_n)^e}{e!}$$

$$=\sum_{e=1}^{\infty}\left(\frac{-nx}{g_n+n+\psi}\right)R_{n+z,e-1}(x)\frac{(-\phi_n)^e}{(e-1)!}+\frac{n+\mu}{g_n+n+\psi}$$

$$=\frac{-nx}{g_n+n+\psi}\sum_{e=1}^{\infty}R_{n+z,e-1}(x)\frac{(-\phi_n)^e}{(e-1)!}+\frac{n+\mu}{g_n+n+\psi}$$

$$=\frac{-nx}{g_n+n+\psi}\sum_{e=1}^{\infty}(-\phi_n)R_{n+z,e-1}(x)\frac{(-\phi_n)^{e-1}}{(e-1)!}+\frac{n+\mu}{g_n+n+\psi}$$

$$= \left(\frac{n\phi_n x}{g_n + n + \psi}\right) \sum_{e=0}^{\infty} R_{n+z,e}(x) \frac{(-\phi_n)^e}{e!}$$

$$+ \frac{n+\mu}{g_n + n + \psi} \qquad [(e \to e+1)\ (\mathrm{n} + z)\ \in\ \mathbb{N}]$$

$$= \frac{n\phi_n x}{g_n + n + \psi} + \frac{n+\mu}{g_n + n + \psi}$$

equality is reached. Similarly

$$\tilde{L}_n(h^2, x) = \sum_{e=0}^{\infty} \left(\frac{e+n+\mu}{g_n + n + \psi}\right)^2 R_{n,e}(x) \frac{(-\phi_n)^e}{e!}$$

$$= \sum_{e=0}^{\infty} \left(\frac{e^2 + 2e(n+\mu) + (n+\mu)^2}{(g_n + n + \psi)^2}\right) R_{n,e}(x) \frac{(-\phi_n)^e}{e!}$$

$$= \sum_{e=0}^{\infty} \left(\frac{e^2}{(g_n + n + \psi)^2}\right) R_{n,e}(x) \frac{(-\phi_n)^e}{e!}$$

$$+ \sum_{e=0}^{\infty} \left(\frac{2e(n+\mu)}{(g_n + n + \psi)^2}\right) R_{n,e}(x) \frac{(-\phi_n)^e}{e!}$$

$$+ \sum_{e=0}^{\infty} \left(\frac{(n+\mu)^2}{(g_n + n + \psi)^2}\right) R_{n,e}(x) \frac{(-\phi_n)^e}{e!}$$

$$= \sum_{e=2}^{\infty} \left(\frac{e(e-1)}{(g_n + n + \psi)^2}\right) R_{n,e}(x) \frac{(-\phi_n)^e}{e!}$$

$$+\sum_{e=1}^{\infty}\left(\frac{e}{(g_n+n+\psi)^2}\right)R_{n,e}(x)\frac{(-\phi_n)^e}{e!}$$

$$+\sum_{e=1}^{\infty}\left(\frac{2(n+\mu)}{(g_n+n+\psi)^2}\right)R_{n,e}(x)\frac{(-\phi_n)^e}{(e-1)!}$$

$$+\sum_{e=0}^{\infty}\left(\frac{(n+\mu)^2}{(g_n+n+\psi)^2}\right)R_{n,e}(x)\frac{(-\phi_n)^e}{e!}$$

$$=\sum_{e=2}^{\infty}\left(\frac{{\phi_n}^2}{(g_n+n+\psi)^2}\right)R_{n,e}(x)\frac{(-\phi_n)^{e-2}}{(e-2)!}$$

$$+\left(\frac{1}{g_n+n+\psi}\right)\left(\frac{n\phi_n x}{g_n+n+\psi}\right)$$

$$+\sum_{e=1}^{\infty}\left(\frac{2(n+\mu)(-nx)}{(g_n+n+\psi)^2}\right)R_{n+z,e-1}(x)\frac{(-\phi_n)^e}{(e-1)!}+\frac{(n+\mu)^2}{(g_n+n+\psi)^2}$$

$$=\sum_{e=2}^{\infty}\left(\frac{{\phi_n}^2 n(n+z)x^2}{(g_n+n+\psi)^2}\right)R_{n+2z,e-2}(x)\frac{(-\phi_n)^{e-2}}{(e-2)!}$$

$$+\left(\frac{1}{g_n+n+\psi}\right)\left(\frac{n\phi_n x}{g_n+n+\psi}\right)$$

$$+\sum_{e=1}^{\infty}\left(\frac{2(n+\mu)\phi_n nx}{(g_n+n+\psi)^2}\right)R_{n+z,e-1}(x)\frac{(-\phi_n)^{e-1}}{(e-1)!}+\frac{(n+\mu)^2}{(g_n+n+\psi)^2}.$$

$$\tilde{L}_n(h^2,x) = \sum_{e=0}^{\infty}\left(\frac{{\phi_n}^2 n(n+z)x^2}{(g_n+n+\psi)^2}\right)R_{n+2z,e}(x)\frac{(-\phi_n)^e}{e!}$$

$$+\left(\frac{1}{g_n+n+\psi}\right)\left(\frac{n\phi_n x}{g_n+n+\psi}\right)$$

$$+\sum_{e=0}^{\infty}\left(\frac{2(n+\mu)\phi_n nx}{(g_n+n+\psi)^2}\right)R_{n+z,e}(x)\frac{(-\phi_n)^e}{e!}+\frac{(n+\mu)^2}{(g_n+n+\psi)^2}$$

$$= \frac{{\phi_n}^2 n(n+z)x^2}{(g_n+n+\psi)^2}+\frac{n\phi_n x}{(g_n+n+\psi)^2}+\frac{2(n+\mu)\phi_n nx}{(g_n+n+\psi)^2}$$

$$+\frac{(n+\mu)^2}{(g_n+n+\psi)^2}$$

$$= \frac{{\phi_n}^2 n(n+z)}{(g_n+n+\psi)^2}x^2+\frac{n\phi_n(2(n+\mu)+1)}{(g_n+n+\psi)^2}x+\frac{(n+\mu)^2}{(g_n+n+\psi)^2}$$

using $(n+z)\in\mathbb{N}_0$ and $(n+2z)\in\mathbb{N}_0$

$$\tilde{L}_n(h^2,x) = \frac{(\phi_n)^2 n(n+z)}{(g_n+n+\psi)^2}x^2+\frac{(2(n+\mu)+1)n\phi_n}{(g_n+n+\psi)^2}x$$

$$+\frac{(n+\mu)^2}{(g_n+n+\psi)^2}$$

is found. This completes the proof.

Theorem 2.1 Let

$$\tilde{L}_n(f,x) = \sum_{e=0}^{\infty} f\left(\frac{e+n+\mu}{g_n+n+\psi}\right) R_{n,e}(x)\frac{(-\phi_n)^e}{e!}.$$

Then, for all $f \in C\left[0,\frac{n+\mu}{n+\psi}\right]$,

$$\lim_{n\to\infty}\left\|\tilde{L}_n(f,x) - f(x)\right\|_{C\left[0,\frac{n+\mu}{n+\psi}\right]} = 0\,.$$

Proof It is sufficient to clearly show that the conditions of the Korovkin's Theorem are satisfied. According to Lemma 2.1, since the equation

$$\left|\tilde{L}_n(1,x) - 1\right| = \left|\sum_{e=0}^{\infty} R_{n,e}(x)\frac{(-\phi_n)^e}{e!} - 1\right| = 0$$

is valid, according to the norm definition in $C\left[0,\frac{n+\mu}{n+\psi}\right]$;

$$\lim_{n\to\infty}\left\|\tilde{L}_n(1,x) - 1\right\|_{C\left[0,\frac{n+\mu}{n+\psi}\right]} = 0\,.$$

is found. Similarly since the equality

$$\left|\tilde{L}_n(h,x) - x\right| = \left|\frac{n\phi_n x}{g_n+n+\psi} + \frac{n+\mu}{g_n+n+\psi} - x\right|$$

$$= \left|x\left(\frac{\phi_n}{g_n+n+\psi}n - 1\right) + \frac{n+\mu}{g_n+n+\psi}\right|$$

is valid. For $x \in \left[0,\frac{n+\mu}{n+\psi}\right]$ and $\lim_{n\to\infty}\frac{\phi_n}{g_n+n+\psi}n = 1$, the inequality

$$\max_{x\in\left[0,\frac{n+\mu}{n+\psi}\right]}\left|\tilde{L}_n(h,x)-x\right|$$

$$=\max_{x\in\left[0,\frac{n+\mu}{n+\psi}\right]}\left|x\left(\frac{\phi_n}{g_n+n+\psi}n-1\right)+\frac{n+\mu}{g_n+n+\psi}\right|$$

$$\leq\left|\frac{n+\mu}{n+\psi}\right|\left|\frac{\phi_n}{g_n+n+\psi}n-1\right|+\left|\frac{n+\mu}{g_n+n+\psi}\right|.$$

can be written. If the limit of both sides is taken

$$\lim_{n\to\infty}\left\|\tilde{L}_n(h,x)-x\right\|_{C\left[0,\frac{n+\mu}{n+\psi}\right]}$$

$$\leq\lim_{n\to\infty}\left|\frac{n+\mu}{n+\psi}\right|\ \lim_{n\to\infty}\left|\frac{n\phi_n}{g_n+n+\psi}-1\right|+\lim_{n\to\infty}\left|\frac{n+\mu}{g_n+n+\psi}\right|=0$$

and

$$\lim_{n\to\infty}\left\|\tilde{L}_n(h,x)-x\right\|_{C\left[0,\frac{n+\mu}{n+\psi}\right]}=0.$$

is found. Finally

$$\lim_{n\to\infty}\left\|\tilde{L}_n(h^2,x)-x^2\right\|_{C\left[0,\frac{n+\mu}{n+\psi}\right]}=0.$$

will be shown.

$$\left|\tilde{L}_n(h^2,x)-x^2\right|$$

$$=\left|\frac{(\phi_n)^2n(n+z)}{(g_n+n+\psi)^2}x^2+\frac{(2(n+\mu)+1)n\phi_n}{(g_n+n+\psi)^2}x\right.$$

$$+\frac{(n+\mu)^2}{(g_n+n+\psi)^2}-x^2\Bigg|$$

$$=\Bigg|x^2\left(\left(\frac{\phi_n}{g_n+n+\psi}\right)^2 n(n+z)-1\right)$$

$$+x\left(\frac{(2(n+\mu)+1)n\phi_n}{(g_n+n+\psi)^2}\right)+\frac{(n+\mu)^2}{(g_n+n+\psi)^2}\Bigg|$$

and then

$$\max_{x\in\left[0,\frac{n+\mu}{n+\psi}\right]}\left|\tilde{L}_n(h^2,x)-x^2\right|$$

$$=\max_{x\in\left[0,\frac{n+\mu}{n+\psi}\right]}\Bigg|x^2\left(\left(\frac{\phi_n}{g_n+n+\psi}\right)^2 n(n+z)-1\right)$$

$$+x\left(\frac{(2(n+\mu)+1)n\phi_n}{(g_n+n+\psi)^2}\right)+\frac{(n+\mu)^2}{(g_n+n+\psi)^2}\Bigg|$$

$$\leq\left(\frac{n+\mu}{n+\psi}\right)^2\left|\left(\left(\frac{\phi_n}{g_n+n+\psi}\right)^2 n(n+z)-1\right)\right|$$

$$+\left(\frac{n+\mu}{n+\psi}\right)\left|\left(\frac{(2(n+\mu)+1)n\phi_n}{(g_n+n+\psi)^2}\right)\right|+\left|\frac{(n+\mu)^2}{(g_n+n+\psi)^2}\right|$$

is found. Since the definitions of (ϕ_n), (g_n) and using $(n+z)\in\mathbb{N}$

$$\lim_{n\to\infty}\left(\frac{\phi_n}{g_n+n+\psi}\right)^2 n(n+z)$$

$$= \lim_{n\to\infty}\left(\frac{\phi_n}{g_n+n+\psi}\right) n \lim_{n\to\infty}\left(\frac{\phi_n}{g_n+n+\psi}\right)(n+z) = 1,$$

$$\lim_{n\to\infty}\left(\frac{(2(n+\mu)+1)n\phi_n}{(g_n+n+\psi)^2}\right)$$

$$= \lim_{n\to\infty}\left(\frac{(2(n+\mu)+1)}{(g_n+n+\psi)}\right)\lim_{n\to\infty}\left(\frac{n\phi_n}{g_n+n+\psi}\right) = 0$$

equations are valid. So

$$\lim_{n\to\infty}\left\|\tilde{L}_n(h^2,x)-x^2\right\|_{C\left[0,\frac{n+\mu}{n+\psi}\right]}$$

$$\leq \lim_{n\to\infty}\left(\frac{n+\mu}{n+\psi}\right)^2\left(\left(\frac{\phi_n}{g_n+n+\psi}\right)^2 n(n+z)-1\right)$$

$$+\lim_{n\to\infty}\left(\frac{n+\mu}{n+\psi}\right)\left(\frac{(2(n+\mu)+1)n\phi_n}{(g_n+n+\psi)^2}\right)+\lim_{n\to\infty}\frac{(n+\mu)^2}{(g_n+n+\psi)^2}$$

Since the inequality is valid

$$\lim_{n\to\infty}\left\|\tilde{L}_n(h^2,x)-x^2\right\|_{C\left[0,\frac{n+\mu}{n+\psi}\right]} = 0$$

will be shown. Then, since all the conditions of Korovkin's Theorem are valid, for every $f \in C\left[0,\frac{n+\mu}{n+\psi}\right]$

$$\lim_{n\to\infty}\left\|\tilde{L}_n(f,x)-f(x)\right\|_{C\left[0,\frac{n+\mu}{n+\psi}\right]} = 0$$

is achieved.

RESULTS

In this section, to analyze the results of the method is shown in this study, the approximation rate of the generalized Gadjiev Ibragimov operator in the space of continuous functions on $\left[0, \frac{n+\mu}{n+\psi}\right]$ will be calculated with the help of the modulus of continuity. First of all, the definition of modulus of continuity will be reminded.

Let $K = \left[0, \frac{n+\mu}{n+\psi}\right]$, for all $f \in C(K), \delta > 0$ modulus of continuity of function f as defined

$$\omega(f, \delta) = \sup_{\substack{|t-x|<\delta \\ x \in K}} |f(t) - f(x)|.$$

Here, the following known features of the modulus of continuity will be used.

Let $f \in C(K)$.

i) Let $\delta \geq 0$. $\omega(f, \delta)$ is a monotonically increasing function with respect to δ.

ii) For all $f \in C(K)$, $\lim_{\delta \to 0} \omega(f, \delta) = 0$.

iii) For $\lambda > 0$, $\omega(f, \lambda\delta) \leq (1 + \lambda)\omega(f, \delta)$.

iv) Let $x, t \in B$ and for all $f \in C(K)$,

$$|f(t)-f(x)| \leq \omega(f,\delta)\left(1+\frac{|t-x|}{\delta}\right).$$

Theorem 3.1 Let $f \in C(K)$ and (ϕ_n), (g_n) be sequences defined in Definition 2.1. In this case, for n is large enough and D is a constant where independent from (ϕ_n), (g_n);

$$\left\|\tilde{L}_n(f,x)-f(x)\right\|_{C\left[0,\frac{n+\mu}{n+\psi}\right]}$$

$$\leq D\omega\left(f;\sqrt{\left(n\frac{\phi_n}{g_n+n+\psi}-1\right)^2+\frac{2\phi_n}{g_n+n+\psi}z+5}\right).$$

Proof

Using

$$\left|\tilde{L}_n(f,x)-f(x)\right| \leq \tilde{L}_n(|f(t)-f(x)|,x),$$

we get

$$\left|\tilde{L}_n(f,x)-f(x)\right| \leq \sum_{e=0}^{\infty}\left|f\left(\frac{e+n+\mu}{g_n+n+\psi}\right)-f(x)\right| R_{n,e}(x)\frac{(-\phi_n)^e}{e!}$$

If $h=\frac{e+n+\mu}{g_n+n+\psi}$is selected in properties of modulus of continuity; for all $\delta_n > 0$

$$\left|f\left(\frac{e+n+\mu}{g_n+n+\psi}\right)-f(x)\right| \leq \omega(f,\delta_n)\left(1+\frac{\left|\frac{e+n+\mu}{g_n+n+\psi}-x\right|}{\delta_n}\right)$$

is written. Using linearity and positivity, it is found as

$$\left|\tilde{L}_n(f,x)-f(x)\right|$$

$$\leq \sum_{e=0}^{\infty}\omega(f,\delta_n)\left(1+\frac{\left|\frac{e+n+\mu}{g_n+n+\psi}-x\right|}{\delta_n}\right)R_{n,e}(x)\frac{(-\phi_n)^e}{e!}$$

$$=\omega(f,\delta_n)\left\{\frac{1}{\delta_n}\sum_{e=0}^{\infty}\left|\left(\frac{e+n+\mu}{g_n+n+\psi}\right)-x\right|R_{n,e}(x)\frac{(-\phi_n)^e}{e!}+1\right\}$$

Here, if M is defined as

$$M=\sum_{e=0}^{\infty}\left|\left(\frac{e+n+\mu}{g_n+n+\psi}\right)-x\right|R_{n,e}(x)\frac{(-\phi_n)^e}{e!}$$

$$=\sum_{e=0}^{\infty}\left(\left|\left(\frac{e+n+\mu}{g_n+n+\psi}\right)-x\right|^2\right)^{\frac{1}{2}}\left[R_{n,e}(x)\frac{(-\phi_n)^e}{e!}\right]^{\frac{1}{2}}$$

$$\times\left[R_{n,e}(x)\frac{(-\phi_n)^e}{e!}\right]^{1/2}.$$

From the Cauchy Schwarz inequality, it can be written as

$$M \leq \left[\sum_{e=0}^{\infty}\left|\left(\frac{e+n+\mu}{g_n+n+\psi}\right)-x\right|^2 R_{n,e}(x)\frac{(-\phi_n)^e}{e!}\right]^{\frac{1}{2}}$$

$$\times\left[R_{n,e}(x)\frac{(-\phi_n)^e}{e!}\right]^{1/2}$$

$$=\left[\sum_{e=0}^{\infty}\left|\left(\frac{e+n+\mu}{g_n+n+\psi}\right)-x\right|^2 R_{n,e}(x)\frac{(-\phi_n)^e}{e!}\right]^{1/2}$$

Therefore, since

$$\left|\tilde{L}_n(f,x)-f(x)\right|$$

$$\leq \omega(f,\delta_n)\left\{\frac{1}{\delta_n}\left[\sum_{e=0}^{\infty}\left|\frac{e+n+\mu}{g_n+n+\psi}-x\right|^2 R_{n,e}(x)\frac{(-\phi_n)^e}{e!}\right]^{1/2}+1\right\}$$

here using

$$\left(\left(\frac{e+n+\mu}{g_n+n+\psi}\right)-x\right)^2=\left(\frac{e+n+\mu}{g_n+n+\psi}\right)^2-2x\left(\frac{e+n+\mu}{g_n+n+\psi}\right)+x^2,$$

$$\left|\tilde{L}_n(f,x)-f(x)\right| \leq \omega(f,\delta_n)\ \left\{\frac{1}{\delta_n}\left[\sum_{e=0}^{\infty}\left(\frac{e+n+\mu}{g_n+n+\psi}\right)^2 R_{n,e}(x)\frac{(-\phi_n)^e}{e!}\right.\right.$$

$$-2x\sum_{e=0}^{\infty}\left(\frac{e+n+\mu}{g_n+n+\psi}\right)R_{n,e}(x)\frac{(-\phi_n)^e}{e!}$$

$$+x^2\sum_{e=0}^{\infty} R_{n,e}(x)\frac{(-\phi_n)^e}{e!}\Bigg]^{1/2}+1\Bigg\}$$

$$= \omega(f,\delta_n)\left\{\frac{1}{\delta_n}\left(\tilde{L}_n(h^2,x)-2x\tilde{L}_n(\mathrm{h},x)+x^2\tilde{L}_n(1,x)\right)^{1/2}+1\right\}$$

inequality is written. For $x\in\left[0,\frac{n+\mu}{n+\psi}\right]$ If $\tilde{L}_n(h^2,x),\tilde{L}_n(\mathrm{h},x),\tilde{L}_n(1,x)$ are written in this inequality. Considering definition of (ϕ_n), (g_n), for n large enough $\frac{\phi_n}{g_n+n+\psi}\leq 1,\frac{\phi_n}{g_n+n+\psi}n\leq 2$ and using $\frac{2\mu n(\phi_n)^2}{(g_n+n+\psi)^2}<\frac{4\mu}{g_n+n+\psi}<\frac{4\mu}{n+\psi}$ and $\frac{n(\phi_n)^2}{(g_n+n+\psi)^2}<\frac{2}{g_n+n+\psi}<\frac{2}{n+\psi}$ we get

$$\left|\tilde{L}_n(f,x)-f(x)\right|$$

$$\leq \omega(f,\delta_n)\left\{\frac{1}{\delta_n}\left(\tilde{L}_n(h^2,x)-2x\tilde{L}_n(\mathrm{h},x)+x^2\tilde{L}_n(1,x)\right)^{1/2}+1\right\}$$

$$\leq \omega(f,\delta_n)\left\{\frac{1}{\delta_n}\left[\left(\frac{n+\mu}{n+\psi}\right)^2\left(\frac{\phi_n}{g_n+n+\psi}\right)^2 n(n+z)\right.\right.$$

$$+\left(\frac{n+\mu}{n+\psi}\right)\left(\frac{(2(n+\mu)+1)n\phi_n}{(g_n+n+\psi)^2}\right)+\frac{(n+\mu)^2}{(g_n+n+\psi)^2}$$

$$-2\left(\frac{n+\mu}{n+\psi}\right)\left(\frac{n\phi_n}{g_n+n+\psi}\left(\frac{n+\mu}{n+\psi}\right)+\frac{n+\mu}{g_n+n+\psi}\right)$$

$$+\left(\frac{n+\mu}{n+\psi}\right)^2\Bigg]^{1/2}+1\Bigg\}$$

$$= \omega(f,\delta_n)\left\{\frac{1}{\delta_n}\left[\left(\frac{n+\mu}{n+\psi}\right)^2\left(\left(\frac{\phi_n}{g_n+n+\psi}\right)^2 n(n+z)\right.\right.\right.$$

$$\left.-2\frac{n\phi_n}{g_n+n+\psi}+1\right)$$

$$+\left(\frac{n+\mu}{n+\psi}\right)\left(\frac{(2(n+\mu)+1)n\phi_n}{(g_n+n+\psi)^2}-2\frac{n+\mu}{g_n+n+\psi}\right)$$

$$\left.\left.+\left(\frac{n+\mu}{g_n+n+\psi}\right)^2\right]^{1/2}+1\right\}$$

$$=\omega(f,\delta_n)\left\{\frac{1}{\delta_n}\left[\left(\frac{n+\mu}{n+\psi}\right)^2\left(\left(\frac{n\phi_n}{g_n+n+\psi}\right)^2-2\frac{n\phi_n}{g_n+n+\psi}+1\right)\right.\right.$$

$$+\left(\frac{n+\mu}{n+\psi}\right)^2\left(\left(\frac{\phi_n}{g_n+n+\psi}\right)^2 nz\right.$$

$$\left.+\frac{1}{\left(\frac{n+\mu}{n+\psi}\right)}\left(\frac{(2(n+\mu)+1)\phi_n n}{(g_n+n+\psi)^2}-2\frac{n+\mu}{g_n+n+\psi}\right)\right)$$

$$\left.\left.+\left(\frac{n+\mu}{g_n+n+\psi}\right)^2\right]^{1/2}+1\right\}$$

$$= \omega(f,\delta_n)\left\{\frac{1}{\delta_n}\left[\left(\frac{n+\mu}{n+\psi}\right)^2\left(\left(\frac{n\phi_n}{g_n+n+\psi}-1\right)^2+2\frac{\phi_n}{g_n+n+\psi}z\right.\right.\right.$$

$$+\frac{1}{\left(\frac{n+\mu}{n+\psi}\right)}\left(\frac{2n^2\phi_n}{(g_n+n+\psi)^2}+\frac{2\mu n\phi_n}{(g_n+n+\psi)^2}+\frac{n\phi_n}{(g_n+n+\psi)^2}\right.$$

$$\left.\left.\left.\left.-\frac{2n}{g_n+n+\psi}-\frac{2\mu}{g_n+n+\psi}\right)\right)+\left(\frac{n+\mu}{g_n+n+\psi}\right)^2\right]^{1/2}+1\right\}$$

$$\leq \omega(f,\delta_n)\left\{\frac{\left(\frac{n+\mu}{n+\psi}\right)}{\delta_n}\left[\left(n\frac{\phi_n}{g_n+n+\psi}-1\right)^2+2\frac{\phi_n}{g_n+n+\psi}z\right.\right.$$

$$\left.\left.+\frac{1}{\left(\frac{n+\mu}{n+\psi}\right)}\left(\frac{4\mu}{n+\psi}+\frac{4}{n+\psi}\right)+\left(\frac{n+\mu}{g_n+n+\psi}\right)^2\right]^{1/2}+1\right\}$$

$$\leq \omega(f,\delta_n)\left\{\frac{\left(\frac{n+\mu}{n+\psi}\right)}{\delta_n}\left[\left(n\frac{\phi_n}{g_n+n+\psi}-1\right)^2\right.\right.$$

$$\left.\left.+2\frac{\phi_n}{g_n+n+\psi}z+5\right]^{\frac{1}{2}}+1\right\}.$$

Then, with selection

$$\delta_n := \sqrt{\left(n\frac{\phi_n}{g_n+n+\psi}-1\right)^2 + \frac{2\phi_n}{g_n+n+\psi}z+5}$$

for independent constant D from n, we have

$$\left\|\tilde{L}_n(f,x)-f(x)\right\|_{C\left[0,\frac{n+\mu}{n+\psi}\right]}$$

$$\leq D\omega\left(f;\sqrt{\left(n\frac{\phi_n}{g_n+n+\psi}-1\right)^2+\frac{2\phi_n}{g_n+n+\psi}z+5}\right).$$

It has been shown by this theorem that the rate of approximation is $\sqrt{\left(n\frac{\phi_n}{g_n+n+\psi}-1\right)^2+\frac{2\phi_n}{g_n+n+\psi}z+5}$ and this rate can be increased according to the selection of ϕ_n and g_n.

Lemma 3.1 The first three central moments for the defined Gadjiev Ibragimov type operator are as follows

i) $\varphi_{n,0}(x)=1,$

ii) $\varphi_{n,1}(x)=x\left[n\frac{\phi_n}{g_n+n+\psi}-1\right]+\frac{n+\mu}{g_n+n+\psi},$

iii) $\varphi_{n,2}(x)=\left(\frac{(\phi_n)^2n(n+z)}{(g_n+n+\psi)^2}-2\frac{n\phi_n}{g_n+n+\psi}+1\right)x^2+\left(\frac{(2(n+\mu)+1)n\phi_n}{(g_n+n+\psi)^2}\right.$

$\left.-2\frac{n+\mu}{g_n+n+\psi}\right)x+\left(\frac{n+\mu}{g_n+n+\psi}\right)^2.$

Proof.

i) $\varphi_{n,0}(x) = \tilde{L}_n((\mathrm{h}-x)^0, x) = \tilde{L}_n(1, x) = 1.$

ii) $\varphi_{n,1}(x) = \tilde{L}_n(\mathrm{h}-x, x) = \tilde{L}_n(\mathrm{h}, x) - x\tilde{L}_n(1, x)$

$$= \frac{n\phi_n}{g_n+n+\psi}x + \frac{n+\mu}{g_n+n+\psi} - x.$$

iii) $\varphi_{n,2}(x) = \tilde{L}_n((\mathrm{h}-x)^2, x)$

$$= \tilde{L}_n(\mathrm{h}^2, x) - 2x\tilde{L}_n(\mathrm{h}, x) + x^2\tilde{L}_n(1, x)$$

$$= \frac{(\phi_n)^2 n(n+z)}{(g_n+n+\psi)^2}x^2 + \left(\frac{(2(n+\mu)+1)n\phi_n}{(g_n+n+\psi)^2}\right)x + \frac{(n+\mu)^2}{(g_n+n+\psi)^2}$$

$$-2x\left(\frac{n\phi_n}{g_n+n+\psi}x + \frac{n+\mu}{g_n+n+\psi}\right) + x^2$$

$$= \left(\frac{(\phi_n)^2 n(n+z)}{(g_n+n+\psi)^2} - 2\frac{n\phi_n}{g_n+n+\psi} + 1\right)x^2$$

$$+\left(\frac{(2(n+\mu)+1)n\phi_n}{(g_n+n+\psi)^2} - 2\frac{n+\mu}{g_n+n+\psi}\right)x + \frac{(n+\mu)^2}{(g_n+n+\psi)^2}.$$

Definition 3.2 Let $P \in]0,1[$, functions that satisfy the condition

$|f(t) - f(x)| \leq N|t-x|^P$

are called Lipschitz class functions. N is called the Lipschitz constant. The class of Lipschitz functions is denoted by $f \in Lip_N\left(P, C\left[0, \frac{n+\mu}{n+\psi}\right]\right)$.

Theorem 3.2 Let $x \in \left[0, \frac{n+\mu}{n+\psi}\right]$. For bounded f defined on $\mathbb{R}$, $f \in Lip_N\left(P, C\left[0, \frac{n+\mu}{n+\psi}\right]\right)$ and $0 < P \leq 1$. Then

$$\left|\tilde{L}_n(f,x) - f(x)\right| \leq PN\left(\left(n\frac{\phi_n}{g_n+n+\psi} - 1\right)^2 + \frac{2\phi_n}{g_n+n+\psi}z + 5\right)^{\frac{P}{2}}.$$

Proof

Let $f \in Lip_N(P)$. Using modulus of continuity we get $\omega(f,\delta) \leq N\delta^P$. From Theorem 3.1

$$\left|\tilde{L}_n(f,x) - f(x)\right|$$

$$\leq P\omega\left(f, \sqrt{\left(n\frac{\phi_n}{g_n+n+\psi} - 1\right)^2 + \frac{2\phi_n}{g_n+n+\psi}z + 5}\right)$$

$$\leq PN\left[\left(n\frac{\phi_n}{g_n+n+\psi} - 1\right)^2 + \frac{2\phi_n}{g_n+n+\psi}z + 5\right]^{P/2}$$

is written. This completes the proof.

Example 3.1 Below is the graph of the approximation of the function $f(x) = x^2$ (dark blue) with the operator $\tilde{L}_n(f,x)$ in Fig.1. Here $(\phi_n) = n$, $(g_n) = n^2$, $\mu = 3$, $\psi = 5$, $\tilde{L}_5(f,x)$ with red, $\tilde{L}_{10}(f,x)$ with black, $\tilde{L}_{12}(f,x)$with cyan, $\tilde{L}_{13}(f,x)$ with magenta drawing is made.

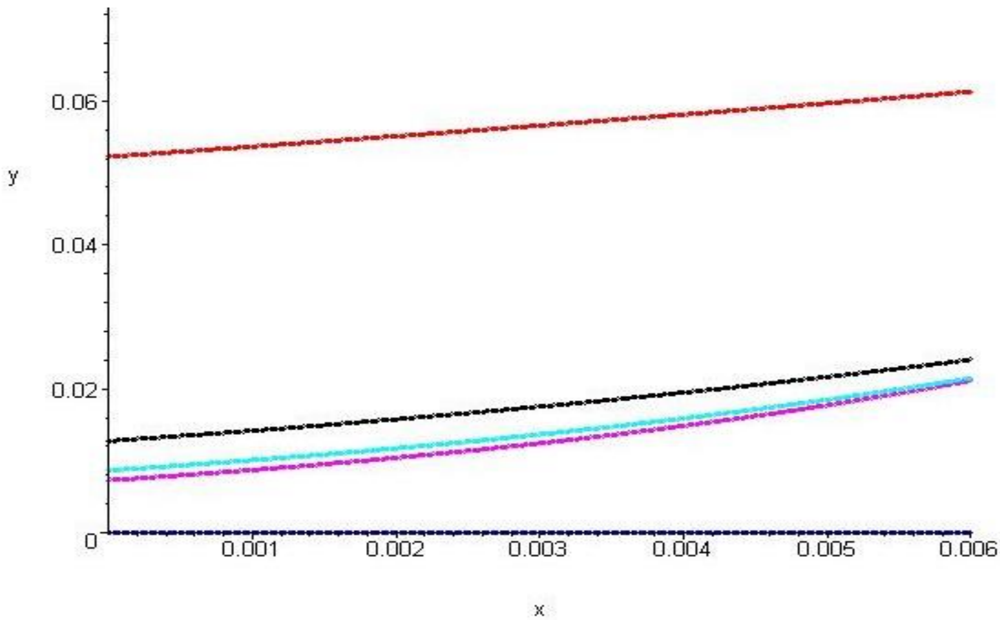

Fig.1. *Graph of approximation to function f*

Example 3.2 Below is the graph of the approximation of the function $f(x) = x^2$ (blue) with the operator $\tilde{L}_n(f,x)$ in Fig.2. Here $(\phi_n) = n$, $(g_n) = n^2$, $\mu = 2$, $\psi = 10$, $\tilde{L}_5(f,x)$ with red, $\tilde{L}_{10}(f,x)$ with black, $\tilde{L}_{12}(f,x)$with cyan, $\tilde{L}_{13}(f,x)$ with magenta drawing is made.

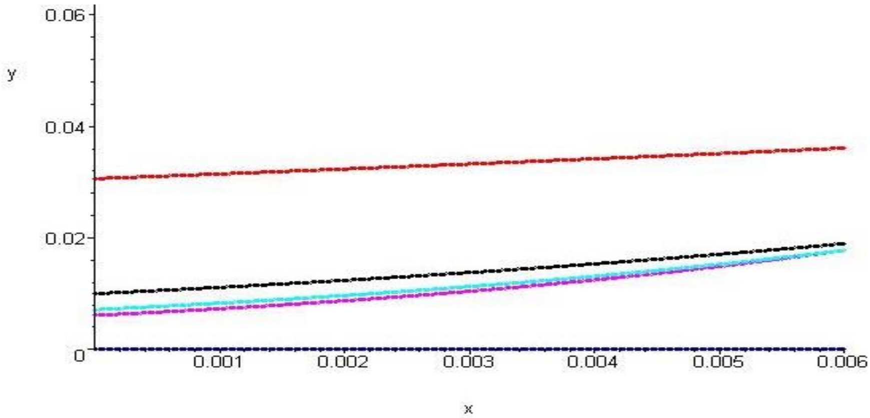

Fig.2. *Graph of approximation to function f*

Example 3.3 Below is the graph of the approximation of the function $f(x) = 5x^2$ (blue) with the operator $\tilde{L}_n(f,x)$ in Fig.3. Here $(\phi_n) = 1$, $(g_n) = n$, $\mu = 1$, $\psi = 1.4$, $\tilde{L}_1(f,x)$ with red, $\tilde{L}_5(f,x)$ with black, $\tilde{L}_{10}(f,x)$with cyan, $\tilde{L}_{15}(f,x)$ with magenta drawing is made.

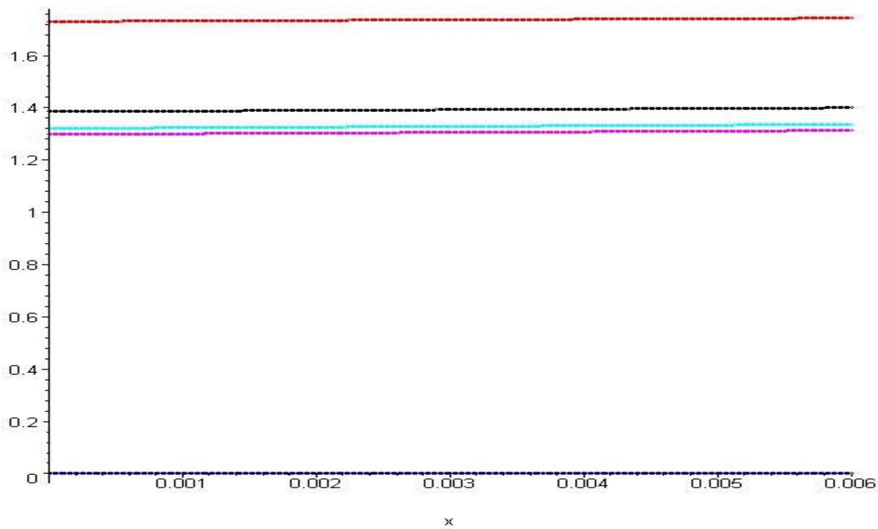

Fig.3 *Graph of approximation to function f*

Example 3.4 Below is the graph of the approximation of the function $f(x) = 5x^2$ (blue) with the operator $\tilde{L}_n(f,x)$ in Fig.4. Here $(\phi_n) = 1$, $(g_n) = n$, $\mu = 2$, $\psi = 3$, $\tilde{L}_1(f,x)$ with red, $\tilde{L}_5(f,x)$ with black, $\tilde{L}_{10}(f,x)$with cyan, $\tilde{L}_{15}(f,x)$ with magenta drawing is made.

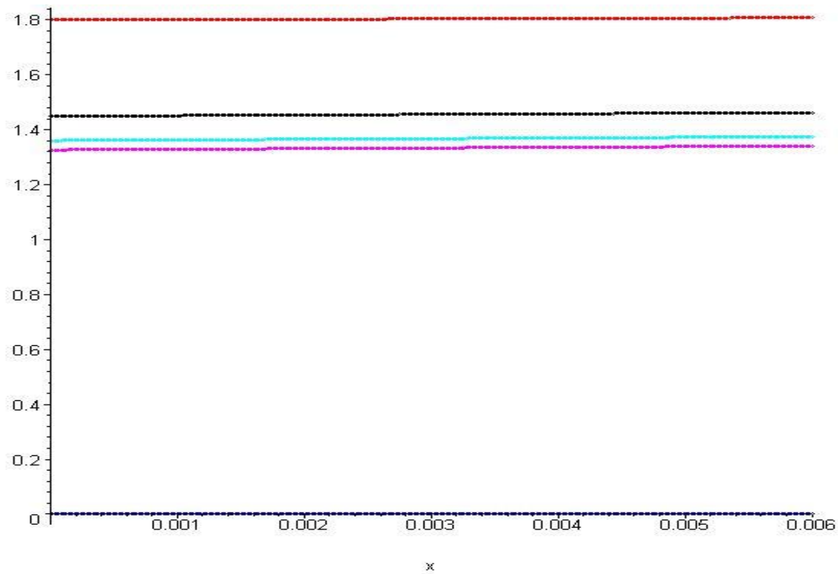

Fig.4 graph of approximation to function f

Example 3.5

Below is the table for the rate of convergence to the function $f(x) = 5e^{(x^2+1)}$ with the operator $\tilde{L}_n(f,x)$.

Table 1. The error bound of function $f(x) = 5e^{(x^2+1)}$ for $(\phi_n) = 1$, $(g_n) = n$.

n	Error calculation of approximation to the function with $\widetilde{G}_n(f,x)$
	$0.2025819086\ 10^8$
10^2	0.144099965310^8
10^3	0.138300075710^8
10^4	0.137721496510^8
10^5	0.137663654310^8
10^6	0.137657870010^8
10^7	0.137657292910^8
10^8	0.137657234010^8
10^9	0.137657229410^8

Example 3.6

Below is the table for the rate of convergence to the function $f(x) = \frac{x^2+1}{3e^{x+1}}$ with the operator $\tilde{L}_n(f,x)$.

***Table 2**. The error bound of function $f(x) = \frac{x^2+1}{3e^{x+1}}$ for $(\phi_n) = 1$, $(g_n) = n$.*

n	Error calculation of approximation to the function with $\widetilde{G}_n(f,x)$
	0.9267182048
10^2	0.9051598220
10^3	0.9025763898
10^4	0.9023129756
10^5	0.9022865822
10^6	0.9022839432
10^7	0.9022836790
10^8	0.9022836524
10^9	0.9022836504

CONCLUSIONS

In our study, where important approximation features were obtained, the selection of the moving range increased the usability of our operator for researches where it was insufficient to deal with a fixed interval. This study, which includes the generalization of some previously defined operators, includes the range limits associated with the establishment of the operator and the changing and generalizing approach features accordingly. For example (Bozma and Bars, 2022) and (Bilgin and Bars, 2022). Since it does not contain derivative expressions as in the classical Gadjiev-Ibragimov operator, important properties related to the operator can be obtained by using only continuous functions and the properties of the sum formula. The described operator is a preferable tool for researchers looking for a suitable operator for daily life problems.

REFERENCES

1. Acar, T. “Rate of convergence for Ibragimov-Gadjiev-Durrmeyer operators”. Demonstratio Mathematica, 50(1), 2017, pp.119-129.
2. Bernstein, S. N. “Sur la représentation des polynômes positifs”. Сообщенія Харьковскаго математическаго общества,14(5), 1915, pp.227-228.
3. Bilgin, N. G., and Cetinkaya, M. “Approximation By Three-Dimensional q-Bernstein-Chlodowsky Polynomials”. Sakarya University Journal of Science Institute, 22(6), 2018, pp.1774-1786.
4. Bilgin, N. G., and Eren. M., A Generalization of Two Dimensional Bernstein-Stancu Operators. Sinop University Journal of Science, 6(2), 2021, pp.130-142.
5. Bozma G. and Bars E. On The Approximation With An Ibragimov-Gadjiev Type Operator, Euroasia Journal Of Mathematics, Engineering, Natural & Medical Sciences International Indexed And Refereed, 9(20), 2022, pp.74-83.
6. Bilgin, N. G., and & Bars E. Effect of Range Variation on Operator Generalization and Approximation Features. Research and Evaluations in Science & Mathematics, 2022/12, 333-354.
7. Bozma, G., & Bars, E. (2022). On the Approximation Properties of an Ibragimov-Gadjiev Type Operator. Euroasia Journal Of Mathematics, Engineering, Natural & Medical Sciences, 9(20), 74_83. Https://Doi.Org/10.38065/Euroasiaorg.932
8. Coskun, T. “On a Construction of Positive Linear Operators for Approximation of Continuous Functions in the Weighted Spaces”. Journal of Computational Analysis & Applications, 13(4). 2011.

9. Deniz, E. and Aral, A. “Convergence properties of Ibragimov-Gadjiev-Durrmeyer operators”. Creat. Math. Inform. 24(1), 2015, pp.17-26

10. Dogru, O. “On a certain family of linear positive operators”. Turkish Journal of Mathematics, 21(4), 1997, pp.387-399.

11. Ghorbanalizadeh, A. “On the order of weighted approximation of unbounded functions”.43rd Annual Iranian Mathematics Conference, 2012, 27-30 August 2012 University of Tabriz.

12. Gonul, N. and Coskun, E. “Weighted approximation by positive linear operators”. Proceedings of IMM of NAS of Azerbaijan, 36, 2012, pp.41-50.

13. Gonul, N. and Coskun, E. “Approximation with modified Gadjiev-Ibragimov operators in C [0, A]”. Journal of Computational Analysis & Applications, 15(1), 2013, pp.868-879.

14. Gonul Bilgin, N. and Coşkun, N. “Comparison Result of Some Gadjiev Ibragimov Type Operators”. Karaelmas Science and Engineering Journal,8(1), 2018, pp.188-196.

15. Gonul Bilgin, N. and Ozgur, N. “Approximation by Two Dimensional Gadjiev-Ibragimov Type Operators”. Ikonion Journal of Mathematics, 1(1), 2019, pp.1-10.

16. Herdem, S. and Buyukyazici, I. “Weighted approximation by q-Ibragimov-Gadjiev operators”. Mathematical Communications, 25(2), 2020, pp.201-212.

17. Ibragimov, I. I. and Gadjiev, A. D. “On a certain Family of linear positive operators”. Soviet Math. Dokl., English Trans, 11, 1970, pp.1092-1095.

18. Kaya, Y., & Gonul, N. (2013, January). A generalization of lacunary equistatistical convergence of positive linear operators. In Abstract and Applied Analysis (Vol. 2013). Hindawi.

CHAPTER XIII

Rough Statıstıcal Convergence For Trıple Dıfference Sequences In Neutrosophıc Normed Spaces

Nazmiye GONUL BILGIN

1.INTRODUCTION

We have tried to cope with uncertainties at many stages of our lives. In addition to situations of truth and falsehood, we frequently encounter situations involving uncertainty, doubt and indecision. It is not easy to say for sure whether some objects are inside or outside a category. In such cases, we can overcome the situation by using partial or gradual membership. All these concepts revealed neutrosophic philosophy, which is considered a newly location of philosophy. On the other hand, the Neutrosophic concept, which was transferred to these fields due to the classical concept of probability being insufficient for mathematics and statistics, took the studies in the field of mathematics one step further. In fact, the concepts of fuzzy and intuitionistic are naturally included in the definition of neutrosophic. After Cantor's set system, was generalized first with fuzzy sets by Zadeh and then with intuitionistic sets by Atanassov, a generalization was made as neutrosophic sets by Smarandache. You can refer to the sources (Kausar et. al, 2023), (Bilgin and Bozma, 2020) for studies on fuzzy and (for studies on intuitionistic fuzzy sets Bilgin, and Bozma, 2021), (Malik and Akram, 2018). Statistical convergence on neutrosophic normed space was first introduced by (Kirisci and Şimşek, 2020). Very soon after (Granados and Dhital, 2021) Neutrosophic statistical convergence is defined using double-indexed sequences in normed space. Neutrosophic triple normed space is presented by (Şahin and Kargın, 2017). Then, many kinds of convergence were transferred to Neutrosophic normed space. Since the subject of convergence has an important place in both mathematics and daily life problems, a lot of studies have been carried out since 2017 on the types of convergence on neutrosophic normed spaces. One is rough convergence.

This type of convergence, which is thought to be helpful in checking the accuracy of numerical analysis and computer programming solutions and thus has the possibility of being applied in daily life, has taken its place in the literature with the studies carried out by (Phu, 2001) for normed spaces (finite dimensional). Later, this type of convergence was combined with the statistical type of convergence by Aytar, who has studies containing important evaluations about rough convergence from different perspectives. e.g. (Aytar, 2008), (Olmez and Aytar, 2021).

Rough statistical convergence is given for difference sequences in (Demir and Gumus, 2022). On the other hand, Rough statistical convergence of triplet sequences is given in (Debnath and Subramanian, 2017). In (Kisi and Gurdal, 2021), statistical convergence is defined of triplet difference sequences on neutrosophic normed space. Rough statistical convergence is studied using tripled sequences on neutrosophic normed spaces in (Bilgin, 2022).

After combining the notion of rough convergence of triplet sequences with statistical convergence theory, now, the definition of rough statistical convergence of triplet difference sequences on neutrosophic normed space was established to complete the corresponding blank in the published.

2. PRIOR INFORMATION

(t_n) is named to be statistically convergent to t if for all $\varepsilon > 0$,

$$\delta(\{n \in \mathbb{N}: |t_k - t| \geq \varepsilon\}) = 0.$$

Here, the three-dimensional version of natural density is given (see e.g Sahiner et all, 2007) as:

$\mathfrak{D} \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is named having a natural density $\delta_3(\mathfrak{D})$, here

$$\delta_3(\mathfrak{D}) = \lim_{p,q,r \to \infty} \frac{|\mathfrak{D}(p,q,r)|}{pqr}$$

exists. Here, $|\mathfrak{D}(p,q,r)|$ demonstrate the numbers of (n_1, n_2, n_3) in $\mathfrak{D}$ where, $p \geq n_1, q \geq n_2 \ and \ r \geq n_3$.

(Sahiner et all, 2007) gave statistical convergence of triple sequence. $(ź_{n_1 n_2 n_3})$ is named to be statistical convergent to ź if for every $\varepsilon > 0$,

$$\delta_3\left(\left\{(n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}: \left|ź_{n_1 n_2 n_3} - ź\right| \geq \varepsilon\right\}\right) = 0.$$

Then, it is shown by $st - \lim_{n_1, n_2, n_3 \to \infty} ź_{n_1 n_2 n_3} = ź$.

Let us recall the definition given by (Debnath and Subramanian, 2017).

$(ź_{n_1 n_2 n_3})$ is called to be rough convergent to ź is shown by $ź_{n_1 n_2 n_3} \xrightarrow{\mathfrak{r}} ź$ such that for every $\varepsilon > 0$, $\exists\, n_0 \in \mathbb{N}: n_1, n_2, n_3 \geq n_0$ then $\left|ź_{n_1 n_2 n_3} - ź\right| < \mathfrak{r} + \varepsilon$.

The rough limit set of $(ź_{n_1 n_2 n_3})$ is demonstrated with $\mathcal{Lim}^{\mathfrak{r}}_{n_1 n_2 n_3} := \left\{ź: ź_{n_1 n_2 n_3} \xrightarrow{\mathfrak{r}} ź\right\}$. It can be easily seen that rough limit set is not unique.

Let us recall some necessary information about the neutrosophic normed space. Let $\mathbb{X} \neq \emptyset$, $\sigma_t(t), \varkappa_u(t)$ and $\beth_f(t)$ are the degrees of truth, falsity and uncertainty. We defined a neutrosophic set $\aleph$ as: For all t in $\mathbb{X}$; $\sigma_t(t), \varkappa_u(t)$ and $\beth_f(t) \in [0,1]$,

$$\aleph = \left\{ \left(t, \sigma_t(t), \varkappa_u(t), \beth_f(t) \right) : t \in \mathbb{X} \right\}, 0 \leq \sigma_t(t) + \varkappa_u(t) + \beth_f(t) \leq 2.$$

Here, it must be pointed out that, $\varkappa_u(t)$ is an independent component, $\sigma_t(t)$ and $\beth_f(t)$ are dependent components. **(**Smarandache, 1998).

Now, we remember definition of Neutrosophic Normed spaces given in (Kirisci and Simsek, 2020).

Let $\mathbb{X}$ be a linear spaces, $\star$ and $\diamond$ demonstrate the continuous $\mathfrak{t}-$ norm and continuous $\mathfrak{t}-$ conorm on $\mathbb{R}$. The notation of Neutrosophic Normed is $\left\{ \left((t, \mathfrak{o}), \sigma_t(t, \mathfrak{o}), \varkappa_u(t, \mathfrak{o}), \beth_f(t, \mathfrak{o}) \right) : (t, \mathfrak{o}) \in \mathbb{X} \times (0, \infty) \right\}$. Here σ_t, $\varkappa_u$ and $\beth_f$ shown the degree of correctness, uncertainty and falsity of $(t, \mathfrak{o})$ on $\mathbb{X} \times (0, \infty)$ fulfills the below criteria: For each $t_1, t_2 \in \mathbb{X}$,

i. For every $\mathfrak{o} \in \mathbb{R}^+$ $\sigma_t(t, \mathfrak{o}) + \varkappa_u(t, \mathfrak{o}) + \beth_f(t, \mathfrak{o}) \leq 2$,
ii. For every $\mathfrak{o}_1, \mathfrak{o}_2 \in \mathbb{R}^+$,
$\sigma_t(t_1, \mathfrak{o}_1) \star \sigma_t(t_2, \mathfrak{o}_2) \leq \sigma_t(t_1 + t_2, \mathfrak{o}_1 + \mathfrak{o}_2)$,
$\varkappa_u(t_1, \mathfrak{o}_1) \diamond \varkappa_u(t_2, \mathfrak{o}_2) \geq \varkappa_u(t_1 + t_2, \mathfrak{o}_1 + \mathfrak{o}_2)$,
$\beth_f(t_1, \mathfrak{o}_1) \diamond \beth_f(t_2, \mathfrak{o}_2) \geq \beth_f(t_1 + t_2, \mathfrak{o}_1 + \mathfrak{o}_2)$.
iii. For every $\mathfrak{p} \in \mathbb{R}^+$, $\sigma_t(t, \mathfrak{o}) = 1 \Leftrightarrow t = 0$, $\varkappa_u(t, \mathfrak{o}) = 0 \Leftrightarrow t = 0$, $\beth_f(t, \mathfrak{o}) = 0 \Leftrightarrow t = 0$,
iv. For each $u \neq 0$, $\sigma_t(u\mathfrak{o}, \mathfrak{p}) = \sigma_t\left(\mathfrak{o}, \frac{\mathfrak{p}}{|u|}\right)$, $\varkappa_u(u\mathfrak{o}, \mathfrak{p}) = \varkappa_u\left(\mathfrak{o}, \frac{\mathfrak{p}}{|u|}\right)$ and $\beth_f(u\mathfrak{o}, \mathfrak{p}) = \beth_f\left(\mathfrak{o}, \frac{\mathfrak{p}}{|u|}\right)$.
v. $\varkappa_u(\mathfrak{o}, \cdot)$ and $\beth_f(\mathfrak{o}, \cdot)$ are continuous non-increasing function and $\sigma_t(\mathfrak{o}, \cdot)$ is continuous non-decreasing function,
vi. $\lim_{s \to \infty} \sigma_t(\mathfrak{o}, \mathfrak{p}) = 1$, $\lim_{s \to \infty} \varkappa_u(\mathfrak{o}, \mathfrak{p}) = 0$ and $\lim_{s \to \infty} \beth_f(\mathfrak{o}, \mathfrak{p}) = 0$.

vii. If $s \leq 0$, $\sigma_t(\mathfrak{o},\mathfrak{p}) = 0$, $\varkappa_u(\mathfrak{o},\mathfrak{p}) = 1$ and $\beth_f(\mathfrak{o},\mathfrak{p}) = 1$.

So, $(\mathcal{X}, \sigma_t, \varkappa_u, \beth_f, \star, \diamond)$ is named Neutrosophic Normed Spaces. Here σ_t and $\varkappa_u$ are interdependent and $\beth_f$ is an independent components.

Let $(\mathcal{X}, \sigma_t, \varkappa_u, \beth_f, \star, \diamond)$ be a Neutrosophic Normed Spaces, $(ź_{n_1 n_2 n_3})$ be a triple sequences. $(ź_{n_1 n_2 n_3})$ is named to be rough convergent to ź for some $\mathbb{r} \in \mathbb{R}^+$ such that each $\varepsilon > 0$, there exits a $n_0 \in \mathbb{N}$ and $\gamma \in (0,1)$:for every $n_1, n_2, n_3 \geq n_0$ if $\sigma_t(ź_{n_1 n_2 n_3} - ź, \mathbb{r} + \varepsilon) > 1 - \gamma$, $\varkappa_u(ź_{n_1 n_2 n_3} - ź, \mathbb{r} + \varepsilon) < \gamma$ and $\beth_f(ź_{n_1 n_2 n_3} - ź, \mathbb{r} + \varepsilon) < \gamma$. Then, it is denote with $\mathbb{r} - \lim_{n_1, n_2, n_3 \to \infty} ź_{n_1 n_2 n_3} = ź$. (Bilgin, 2022).

Let $(ź_{n_1 n_2 n_3})$ be a triple sequences in $(\mathcal{X}, \sigma_t, \varkappa_u, \beth_f, \star, \diamond)$. $(ź_{n_1 n_2 n_3})$ is called to be rough statistically convergent to ź for some $\mathbb{r} \in [0,\infty)$ such that all $\varepsilon > 0$ and $\gamma \in (0,1)$ where

$$\delta_3\Big(\big\{(n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}: \sigma_t(ź_{n_1 n_2 n_3} - ź, \mathbb{r} + \varepsilon) \leq 1 - \gamma \text{ or } \varkappa_u(ź_{n_1 n_2 n_3} - ź, \mathbb{r} + \varepsilon)$$

$$\geq \gamma \text{ and } \beth_f(ź_{n_1 n_2 n_3} - y, \mathbb{r} + \varepsilon) \geq \gamma\big\}\Big) = 0.$$

Afterwards, it is indicated by $st - \mathbb{r} - \lim_{n_1, n_2, n_3 \to \infty} ź_{n_1 n_2 n_3} = ź$. (Bilgin, 2022).

For $\mathbb{r} = 0$, rough statistical convergence consistent matching the statistical convergence on $(\mathcal{X}, \sigma_t, \varkappa_u, \beth_f, \star, \diamond)$. Let rough statistical limit set of $(ź_{n_1 n_2 n_3})$ is denoted with;

$$\mathcal{S}t - \mathbb{r} - LIM(ź_{n_1 n_2 n_3}) = \{ź: st - \mathbb{r} - \lim_{n_1, n_2, n_3 \to \infty} ź_{n_1 n_2 n_3} = ź\}.$$

3. MAIN RESULT

Definition 3.1 Let $(\mathcal{X}, \sigma_t, \varkappa_u, \beth_f, \star, \diamond)$ be a Neutrosophic Normed Spaces, $(ź_{n_1 n_2 n_3})$ be a triple sequences. $(ź_{n_1 n_2 n_3})$ is named to be rough Δ-convergent to ź for some $\mathbb{r} \in \mathbb{R}^+$ such that each $\varepsilon > 0$, there exits a $n_0 \in \mathbb{N}$ and $\gamma \in (0,1)$:for every $n_1, n_2, n_3 \geq n_0$ if

$$\sigma_t\left(\Delta_{ź_{n_1 n_2 n_3}} - ź, \mathbb{r} + \varepsilon\right) > 1 - \gamma \quad , \quad \varkappa_u\left(\Delta_{ź_{n_1 n_2 n_3}} - ź, \mathbb{r} + \varepsilon\right) < \gamma,$$
$$\beth_f\left(\Delta_{ź_{n_1 n_2 n_3}} - ź, \mathbb{r} + \varepsilon\right) < \gamma.$$

for all $n_1, n_2, n_3 \geq n_0$, where $n_1, n_2, n_3 \in \mathbb{N}$,

$$\Delta_{ź_{n_1 n_2 n_3}} = ź_{n_1 n_2 n_3} - ź_{n_1 (n_2+1) n_3} - ź_{n_1 n_2 (n_3+1)} + ź_{n_1 (n_2+1)(n_3+1)} - ź_{(n_1+1) n_2 n_3}$$
$$+ ź_{(n_1+1)(n_2+1) n_3} + ź_{(n_1+1) n_2 (n_3+1)} - ź_{(n_1+1)(n_2+1)(n_3+1)}.$$

In this case, it is denote with $\mathbb{r} - \lim_{n_1, n_2, n_3 \to \infty} \Delta_{ź_{n_1 n_2 n_3}} = ź$.

Definition 3.2 Let $\left(ź_{n_1 n_2 n_3}\right)$ be a triple sequences in $\left(\mathcal{X}, \sigma_t, \varkappa_u, \beth_f, \star, \diamond\right)$. $\left(ź_{n_1 n_2 n_3}\right)$ is called to be rough statistically convergent to ź for some $\mathbb{r} \in [0, \infty)$ such that each $\varepsilon > 0$ and $\gamma \in (0,1)$ if

$$\delta_3\Big(\Big\{(n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \sigma_t\left(ź_{n_1 n_2 n_3} - ź, \mathbb{r} + \varepsilon\right) \leq 1 - \gamma \text{ or } \varkappa_u\left(ź_{n_1 n_2 n_3} - ź, \mathbb{r} + \varepsilon\right)$$
$$\geq \gamma \; and \; \beth_f\left(ź_{n_1 n_2 n_3} - ź, \mathbb{r} + \varepsilon\right) \geq \gamma\Big\}\Big) = 0$$

for all $n_1, n_2, n_3 \geq n_0$, where $n_1, n_2, n_3 \in \mathbb{N}$,

$$\Delta_{ź_{n_1 n_2 n_3}} = ź_{n_1 n_2 n_3} - ź_{n_1 (n_2+1) n_3} - ź_{n_1 n_2 (n_3+1)} + ź_{n_1 (n_2+1)(n_3+1)} - ź_{(n_1+1) n_2 n_3}$$
$$+ ź_{(1+n_1)(1+n_2) n_3} + ź_{(1+n_1) n_2 (n_3+1)} - ź_{(1+n_1)(1+n_2)(n_3+1)}.$$

Then, it is denote with $st - \mathbb{r} - \lim_{n_1, n_2, n_3 \to \infty} \Delta_{ź_{n_1 n_2 n_3}} = ź$.

For $\mathbb{r} = 0$, rough statistical convergence in accordance to statistical convergence on $\left(\mathcal{X}, \sigma_t, \varkappa_u, \beth_f, \star, \diamond\right)$. Let rough statistical limit set of $\left(\Delta_{\acute{z}_{n_1 n_2 n_3}}\right)$ is denoted with;

$$\mathcal{St} - \mathbb{r} - LIM\left(\Delta_{\acute{z}_{n_1 n_2 n_3}}\right) = \left\{\acute{z}: st - \mathbb{r} - \lim_{n_1, n_2, n_3 \to \infty} \Delta_{\acute{z}_{n_1 n_2 n_3}} = \acute{z}\right\}.$$

Definition 3.3 Let $\left(\mathcal{X}, \sigma_t, \varkappa_u, \beth_f, \star, \diamond\right)$ be a Neutrosophic Normed Spaces, $\left(\acute{z}_{n_1 n_2 n_3}\right)$ be a triple sequences. $\left(\acute{z}_{n_1 n_2 n_3}\right)$ is named to be rough statistically bounded for some $\mathbb{r} \in \mathbb{R}^+$ such that each $\varepsilon > 0$ and $\gamma \in (0,1)$ if there exists a $\mathcal{T} > 0$ such that

$$\delta_3\left(\left\{(n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}: \sigma_t\left(\Delta_{\acute{z}_{n_1 n_2 n_3}}, \mathcal{T}\right) \leq 1 - \gamma \text{ or } \varkappa_u\left(\Delta_{\acute{z}_{n_1 n_2 n_3}}, \mathcal{T}\right) \geq \gamma \ \ and \ \ \beth_f\left(\Delta_{\acute{z}_{n_1 n_2 n_3}}, \mathcal{T}\right) \geq \gamma\right\}\right) = 0$$

where $\Delta_{\acute{z}_{n_1 n_2 n_3}} = \acute{z}_{n_1 n_2 n_3} - \acute{z}_{n_1 (n_2+1) n_3} - \acute{z}_{n_1 n_2 (n_3+1)} + \acute{z}_{n_1 (n_2+1)(n_3+1)} - \acute{z}_{(n_1+1) n_2 n_3}$
$+ \acute{z}_{(n_1+1)(n_2+1) n_3} + \acute{z}_{(n_1+1) n_2 (n_3+1)} - \acute{z}_{(n_1+1)(n_2+1)(n_3+1)}$.

Now, using these descriptions, the next significant theorems for triple sequences in Neutrosophic normed spaces can be proved.

Lemma 3.1 Let $\left(\mathcal{X}, \sigma_t, \varkappa_u, \beth_f, \star, \diamond\right)$ be a Neutrosophic Normed Spaces, For a triple sequences $(\acute{z}_{mno})$ and some $\mathbb{r} \geq 0$, if $\mathcal{St} - \mathbb{r} - LIM\left(\Delta_{\acute{z}_{n_1 n_2 n_3}}\right) \neq \emptyset$ then $\left(\acute{z}_{n_1 n_2 n_3}\right)$ is rough bounded sequnces in $\left(\mathcal{X}, \sigma_t, \varkappa_u, \beth_f, \star, \diamond\right)$.

Proof Let $(\acute{z}_{mno})$ be a triple sequences in $\left(\mathcal{X}, \sigma_t, \varkappa_u, \beth_f, \star, \diamond\right)$ and some > 0, $\mathcal{St} - \mathbb{r} - LIM\left(\Delta_{\acute{z}_{n_1 n_2 n_3}}\right) \neq \emptyset$. Then there exists ź and $\acute{z} \in \mathcal{St} - \mathbb{r} - LIM\left(\Delta_{\acute{z}_{n_1 n_2 n_3}}\right)$. For all $\varepsilon > 0$ and $0 < \gamma < 1$, it is written

$$\delta_3\left(\left\{(n_1,n_2,n_3)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\sigma_t\left(\Delta_{\acute{z}_{n_1n_2n_3}}-\acute{z},\mathbb{r}+\varepsilon\right)\right.\right.$$
$$\leq 1-\gamma \text{ or } \varkappa_u\left(\Delta_{\acute{z}_{n_1n_2n_3}}-\acute{z},\mathbb{r}+\varepsilon\right)\geq\gamma$$
$$\left.\left. and\ \beth_f\left(\Delta_{\acute{z}_{n_1n_2n_3}}-\acute{z},\ \mathbb{r}+\varepsilon\right)\geq\gamma\right\}\right)=0.$$

So, $\left(\acute{z}_{n_1n_2n_3}\right)$ is statistically bounded in $\left(\mathcal{X},\sigma_t,\varkappa_u,\beth_f,\ \star,\diamond\right)$.

Lemma 3.2 Let $\left(\mathcal{X},\sigma_t,\varkappa_u,\beth_f,\ \star,\diamond\right)$ be a Neutrosophic Normed Spaces, $\left(\acute{z}_{n_1n_2n_3}\right)$ be a triple sequences. If $\left(\acute{z}_{n_1n_2n_3}\right)$ is rough bounded sequnces in $\left(\mathcal{X},\sigma_t,\varkappa_u,\beth_f,\ \star,\diamond\right)$ then, for some $\mathbb{r}\geq 0$, if $\mathcal{S}t-\mathbb{r}-LIM\left(\Delta_{\acute{z}_{n_1n_2n_3}}\right)\neq\emptyset$.

Proof $\left(\acute{z}_{n_1n_2n_3}\right)$ be a triple sequences in $\left(\mathcal{X},\sigma_t,\varkappa_u,\beth_f,\ \star,\diamond\right)$ and $\left(\acute{z}_{n_1n_2n_3}\right)$ is bounded sequnces. For each $\varepsilon>0$, $\gamma\in(0,1)$ and some $\mathbb{r}\geq 0$ there exists a $\mathcal{T}>0$ such that

$$\delta_3\left(\left\{(n_1,n_2,n_3)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\sigma_t\left(\Delta_{\acute{z}_{n_1n_2n_3}},\mathcal{T}\right)\right.\right.$$
$$\leq 1-\gamma \text{ or } \varkappa_u\left(\Delta_{\acute{z}_{n_1n_2n_3}},\mathcal{T}\right)\geq\gamma$$
$$and\ \beth_f\left(\Delta_{\acute{z}_{n_1n_2n_3}},\ \mathcal{T}\right)\geq\gamma\})=0.$$

Let a set of the form

$$\mathfrak{M}=\left\{(n_1,n_2,n_3)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\sigma_t\left(\Delta_{\acute{z}_{n_1n_2n_3}},\mathcal{T}\right)\leq 1-\right.$$
$$\gamma \text{ or } \varkappa_u\left(\Delta_{\acute{z}_{n_1n_2n_3}},\mathcal{T}\right)\geq\gamma$$
$$\left. and\ \beth_f\left(\Delta_{\acute{z}_{n_1n_2n_3}},\mathcal{T}\right)\geq\gamma\right\}$$

is defined. For $(n_1,n_2,n_3)\in\mathfrak{M}^c$, it is written $\sigma_t\left(\Delta_{\acute{z}_{n_1n_2n_3}},\mathcal{T}\right)>1-\gamma$ and $\varkappa_u\left(\Delta_{\acute{z}_{n_1n_2n_3}},\mathcal{T}\right)<\gamma$, $\beth_f\left(\Delta_{\acute{z}_{n_1n_2n_3}},\mathcal{T}\right)<\gamma$. Furthermore, $\sigma_t\left(\Delta_{\acute{z}_{n_1n_2n_3}},\mathbb{r}+\mathcal{T}\right)>1-\gamma$ and $\varkappa_u\left(\Delta_{\acute{z}_{n_1n_2n_3}},\mathbb{r}+\mathcal{T}\right)<\gamma$, $\beth_f\left(\Delta_{\acute{z}_{n_1n_2n_3}},\mathbb{r}+\mathcal{T}\right)<\gamma$.

So $0 \in \mathcal{S}t - \mathbb{r} - LIM\left(\Delta_{ź_{n_1 n_2 n_3}}\right)$. Hence $\mathcal{S}t - \mathbb{r} - LIM\left(\Delta_{ź_{n_1 n_2 n_3}}\right) \neq \emptyset$.

In the next part, some topological characteristics of the set of $\mathcal{S}t - \mathbb{r} - LIM\left(\Delta_{ź_{n_1 n_2 n_3}}\right)$ will be examined.

Theorem 3.1 Let $\left(\Delta_{ź_{n_1 n_2 n_3}}\right)$ is a triple sequences in $(\mathcal{X}, \sigma_t, \varkappa_u, ℶ_f, \star, \diamond)$ then $\mathcal{S}t - \mathbb{r} - LIM\left(\Delta_{ź_{n_1 n_2 n_3}}\right)$ is closed sets.

Proof It's simple to demonstrate that $\mathcal{S}t - \mathbb{r} - LIM\left(\Delta_{ź_{n_1 n_2 n_3}}\right) = \emptyset$, so let $St - \mathbb{r} - LIM\left(\Delta_{ź_{n_1 n_2 n_3}}\right) \neq \emptyset$. Then, choosing a triple sequences $\left(ź_{n_1 n_2 n_3}\right)$ and $0 < \gamma'$, where $\gamma \diamond \gamma > \gamma', 1 - \gamma' < (1 - \gamma) \star (1 - \gamma)$ and $st - \mathbb{r} - \lim_{n_1, n_2, n_3 \to \infty} \Delta_{ź_{n_1 n_2 n_3}} = ź$. It will be shown that $ź \in \mathcal{S}t - \mathbb{r} - LIM\left(\Delta_{ź_{n_1 n_2 n_3}}\right)$. Let $\varepsilon > 0$ and we use definition; there exists a $n_0 \in \mathbb{N}$ such that for $n_1, n_2, n_3 \geq n_0$,

$\sigma_t\left(\Delta_{ź_{n_1 n_2 n_3}} - ź, \frac{\varepsilon}{3}\right) > 1 - \gamma$, $\varkappa_u\left(\Delta_{ź_{n_1 n_2 n_3}} - ź, \frac{\varepsilon}{3}\right) < \gamma$ and $ℶ_f\left(\Delta_{ź_{n_1 n_2 n_3}} - ź, \frac{\varepsilon}{3}\right) < \gamma$.

If choosing $ź_{m'n'o'} \in \mathcal{S}t - \mathbb{r} - LIM\left(\Delta_{ź_{n_1 n_2 n_3}}\right)$ where $\ddot{n}_1, \ddot{n}_2, \ddot{n}_3 > n_0$ such that

$$\delta_3\Big(\Big\{(n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}: \sigma_t\left(\Delta_{ź_{n_1 n_2 n_3}} - \Delta_{ź_{\ddot{n}_1 \ddot{n}_2 \ddot{n}_3}}, \mathbb{r} + \frac{\varepsilon}{3}\right) \leq 1 - \gamma \text{ or } \varkappa_u\left(\Delta_{ź_{n_1 n_2 n_3}} - \Delta_{ź_{\ddot{n}_1 \ddot{n}_2 \ddot{n}_3}}, \mathbb{r} + \frac{\varepsilon}{3}\right) \geq \gamma$$

$$and\ ℶ_f\left(\Delta_{ź_{n_1 n_2 n_3}} - \Delta_{ź_{\ddot{n}_1 \ddot{n}_2 \ddot{n}_3}}, \mathbb{r} + \frac{\varepsilon}{3}\right) \geq \gamma\Big\}\Big) = 0.$$

For $(a,b,c) \in \left\{(n_1,n_2,n_3) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}: \sigma_t\left(\Delta_{ź_{n_1n_2n_3}} - \Delta_{ź_{\ddot{n}_1\ddot{n}_2\ddot{n}_3}}, \mathbb{r}+\frac{\varepsilon}{3}\right) > 1-\gamma \text{ or } \varkappa_u\left(\Delta_{ź_{n_1n_2n_3}} - \Delta_{ź_{\ddot{n}_1\ddot{n}_2\ddot{n}_3}}, \mathbb{r}+\frac{\varepsilon}{3}\right) < \gamma \; and \; \beth_f\left(\Delta_{ź_{n_1n_2n_3}} - \Delta_{ź_{\ddot{n}_1\ddot{n}_2\ddot{n}_3}}, \mathbb{r}+\frac{\varepsilon}{3}\right) < \gamma\right\}$.

Furthermore,

$1-\gamma' < (1-\gamma)\star(1-\gamma) < \sigma_t\left(\Delta_{ź_{n_1n_2n_3}} - ź, \mathbb{r}+\varepsilon\right)$ and $\varkappa_u\left(\Delta_{ź_{abc}} - ź,\ \mathbb{r}+\varepsilon\right) \le \gamma \diamond \gamma < \gamma'$,

$\beth_f\left(ź_{n_1n_2n_3} - ź, \mathbb{r}+\varepsilon\right) \le \gamma \diamond \gamma < \gamma^{'}$. So,

$$(a,b,c) \in \left\{(n_1,n_2,n_3) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}: \sigma_t\left(\Delta_{ź_{n_1n_2n_3}} - ź, \mathbb{r}+\varepsilon\right) > 1-\gamma \text{ or }\right.$$

$\left.\varkappa_u\left(\Delta_{ź_{n_1n_2n_3}} - ź, \mathbb{r}+\varepsilon\right) < \gamma \; and \; \beth_f\left(\Delta_{ź_{n_1n_2n_3}} - ź, \mathbb{r}+\varepsilon\right) < \gamma\right\}$. Then,

$$\left\{(n_1,n_2,n_3) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}: \sigma_t\left(\Delta_{ź_{n_1n_2n_3}} - \Delta_{ź_{\ddot{n}_1\ddot{n}_2\ddot{n}_3}}, \mathbb{r}+\frac{\varepsilon}{3}\right) > 1-\gamma,\right.$$

$$\left.\varkappa_u\left(\Delta_{ź_{n_1n_2n_3}} - \Delta_{ź_{\ddot{n}_1\ddot{n}_2\ddot{n}_3}},\ \mathbb{r}+\frac{\varepsilon}{3}\right) < \gamma \text{ and } \beth_f\left(\Delta_{ź_{n_1n_2n_3}} - \Delta_{ź_{\ddot{n}_1\ddot{n}_2\ddot{n}_3}}, \mathbb{r}+\frac{\varepsilon}{3}\right) < \gamma\right\}$$

$$\subseteq \left\{(n_1,n_2,n_3) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}: \sigma_t\left(\Delta_{ź_{n_1n_2n_3}} - ź, \mathbb{r}+\frac{\varepsilon}{3}\right) > 1-\gamma, \varkappa_u\left(\Delta_{ź_{n_1n_2n_3}} - y,\ \mathbb{r}+\frac{\varepsilon}{3}\right) < \gamma\right.$$

$\left.\text{and } \beth_f\left(\Delta_{ź_{n_1n_2n_3}} - ź, \mathbb{r}+\frac{\varepsilon}{3}\right) < \gamma\right\}$.

Then,

$$\delta_3\left\{(n_1,n_2,n_3) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}: \sigma_t\left(\Delta_{ź_{n_1n_2n_3}} - ź, \mathbb{r}+\varepsilon\right) \le 1-\gamma \;\; or\right.$$

$$\left.\varkappa_u\left(\Delta_{ź_{n_1n_2n_3}} - ź,\ \mathbb{r}+\varepsilon\right) \ge \gamma \text{ and } \beth_f\left(\Delta_{ź_{n_1n_2n_3}} - ź, \mathbb{r}+\varepsilon\right) \ge \gamma\right\}$$

$$\leq \delta_3 \left\{(n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}: \sigma_t \left(\Delta_{\acute{z}_{n_1 n_2 n_3}} - \Delta_{\acute{z}_{\ddot{n}_1 \ddot{n}_2 \ddot{n}_3}}, \mathbb{r} + \frac{\varepsilon}{3}\right) \leq 1 - \gamma \ or \right.$$
$$\varkappa_u \left(\Delta_{\acute{z}_{n_1 n_2 n_3}} - \Delta_{\acute{z}_{\ddot{n}_1 \ddot{n}_2 \ddot{n}_3}}, \mathbb{r} + \frac{\varepsilon}{3}\right) \geq \gamma \text{ and } \beth_f \left(\Delta_{\acute{z}_{n_1 n_2 n_3}} - \Delta_{\acute{z}_{\ddot{n}_1 \ddot{n}_2 \ddot{n}_3}}, \mathbb{r} + \frac{\varepsilon}{3}\right) \geq \gamma \Big\}.$$

Thus,

$$\delta_3 \left\{ (n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}: \sigma_t \left(\Delta_{\acute{z}_{n_1 n_2 n_3}} - \acute{z}, \mathbb{r} + \varepsilon\right) \leq 1 - \gamma \ \ or \ \ \varkappa_u \left(\Delta_{\acute{z}_{n_1 n_2 n_3}} - \acute{z}, \ \mathbb{r} + \varepsilon\right) \geq \gamma \ \text{ and } \beth_f \left(\Delta_{\acute{z}_{n_1 n_2 n_3}} - \acute{z}, \mathbb{r} + \varepsilon\right) \geq \gamma \right\} = 0.$$

Consequently, it is shown that $\acute{z} \in \mathcal{St} - \mathbb{r} - LIM\left(\Delta_{\acute{z}_{n_1 n_2 n_3}}\right)$.

Definition 3.4 Let $\left(\mathcal{X}, \sigma_t, \varkappa_u, \beth_f, \ \star, \diamond\right)$ be a Neutrosophic Normed Spaces. For some $\mathbb{r} \in [0, \infty)$, every $\varepsilon > 0$ and $\gamma \in (0,1)$,

$$\delta_3 \left\{ (n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}: \sigma_t \left(\Delta_{\acute{z}_{n_1 n_2 n_3}} - \acute{z}, \mathbb{r} + \varepsilon\right) > 1 - \gamma \ \ and \ \ \varkappa_u \left(\Delta_{\acute{z}_{n_1 n_2 n_3}} - \acute{z}, \ \mathbb{r} + \varepsilon\right) < \gamma \ \text{ and } \ \beth_f \left(\Delta_{\acute{z}_{n_1 n_2 n_3}} - z, \mathbb{r} + \varepsilon\right) < \gamma \right\} \neq 0,$$

then, $\acute{z}$ is termed rough statistical cluster point of $\left(\acute{z}_{n_1 n_2 n_3}\right)$. It is denote with $\mathcal{st} - \mathbb{r} - \Delta cls$ point of $\left(\acute{z}_{n_1 n_2 n_3}\right)$. Let $\mathcal{C}^{\mathbb{r}}_{\Delta_{\acute{z}_{n_1 n_2 n_3}}}$ is demonstrated the set of each $\mathcal{st} - \mathbb{r} - \Delta cls$ point of $\left(\acute{z}_{n_1 n_2 n_3}\right)$ in $\left(\mathcal{X}, \sigma_t, \varkappa_u, \beth_f, \ \star, \diamond\right)$.

Theorem 3.2 Let $\left(\mathcal{X}, \sigma_t, \varkappa_u, \beth_f, \ min, \ max\right)$ be a Neutrosophic Normed Spaces, $\left(\acute{z}_{n_1 n_2 n_3}\right)$ be a triple sequences. Then, for some $\mathbb{r} \in [0, \infty)$, every $\varepsilon > 0$ and $\gamma \in (0,1)$ the set $\mathcal{C}^{\mathbb{r}}_{\Delta_{\acute{z}_{n_1 n_2 n_3}}}$ is closed.

Proof Let $\mathcal{C}^{\mathbb{r}}_{\Delta_{\acute{z}_{n_1n_2n_3}}} \neq \emptyset$ be taken as the proof for $\mathcal{C}^{\mathbb{r}}_{\Delta_{\acute{z}_{n_1n_2n_3}}} = \emptyset$ is clear. Now, choosing $(u_{n_1n_2n_3}) \subseteq \mathcal{C}^{\mathbb{r}}_{\Delta_{\acute{z}_{n_1n_2n_3}}}$ and $\mathbb{r} - \lim_{n_1,n_2,n_3\to\infty} u_{n_1n_2n_3} = u$. If $u \in \mathcal{C}^{\mathbb{r}}_{\Delta_{\acute{z}_{n_1n_2n_3}}}$ is proven. If we use definition of rough convergence of sequences, for every $\varepsilon > 0$ and $\gamma \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that for $m, n, o > n_0$, $\sigma_t\left(\Delta_{\acute{z}_{n_1n_2n_3}} - u, \frac{\varepsilon}{3}\right) > 1-\gamma$, $\varkappa_u\left(\Delta_{\acute{z}_{n_1n_2n_3}} - u, \frac{\varepsilon}{3}\right) < \gamma$ and $\beth_f\left(\Delta_{\acute{z}_{n_1n_2n_3}} - u, \frac{\varepsilon}{3}\right) < \gamma$. Choosing $\tilde{n} \in \mathbb{N}$ where $\tilde{n} \geq n_0$. So, $\sigma_t\left(\acute{z}_{\tilde{n}} - u, \frac{\varepsilon}{3}\right) > 1-\gamma, \varkappa_u\left(\acute{z}_{\tilde{n}} - u, \frac{\varepsilon}{3}\right) < \gamma$ and $\beth_f(\acute{z}_{\tilde{n}} - u, \frac{\varepsilon}{3}) < \gamma$. Using $(u_{n_1n_2n_3}) \subseteq \mathcal{C}^{\mathbb{r}}_{\Delta_{\acute{z}_{n_1n_2n_3}}}$, it is written $\acute{z}_{\tilde{n}} \in \mathcal{C}^{\mathbb{r}}_{\Delta_{\acute{z}_{n_1n_2n_3}}}$. So

$\delta_3\ \left(\left\{ (n_1, n_2, n_3) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}: \sigma_t\left(\Delta_{\acute{z}_{n_1n_2n_3}} - \acute{z}_{\tilde{n}}, \mathbb{r} + \frac{\varepsilon}{3}\right) > 1 - \gamma\ ,\ \varkappa_u\left(\Delta_{\acute{z}_{n_1n_2n_3}} - \acute{z}_{\tilde{n}},\ \mathbb{r} + \frac{\varepsilon}{3}\right) < \gamma \text{ and } \beth_f\left(\Delta_{\acute{z}_{n_1n_2n_3}} - \acute{z}_{\tilde{n}},\ \mathbb{r} + \frac{\varepsilon}{3}\right) < \gamma\right\}\right) \neq 0.$

Taking

$$(\widetilde{n_1}, \widetilde{n_2}, \widetilde{n_3}) \in \left\{(n_1, n_2, n_3) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}: \sigma_t\left(\Delta_{\acute{z}_{n_1n_2n_3}} - \acute{z}_{\tilde{n}}, \mathbb{r} + \frac{\varepsilon}{3}\right) > 1-\gamma, \varkappa_u\left(\Delta_{\acute{z}_{n_1n_2n_3}} - \acute{z}_{\tilde{n}},\ \mathbb{r} + \frac{\varepsilon}{3}\right) < \gamma \text{ and } \beth_f\left(\Delta_{\acute{z}_{n_1n_2n_3}} - \acute{z}_{\tilde{n}},\ \mathbb{r} + \frac{\varepsilon}{3}\right) < \gamma\right\}.$$

Hence, it is written that

$\sigma_t\left(\Delta_{\acute{z}_{\widetilde{n_1}\widetilde{n_2}\widetilde{n_3}}} - \acute{z}_{\tilde{n}}, \mathbb{r} + \frac{\varepsilon}{3}\right) > 1-\gamma$, $\varkappa_u\left(\Delta_{\acute{z}_{\widetilde{n_1}\widetilde{n_2}\widetilde{n_3}}} - \acute{z}_{\tilde{n}},\ \mathbb{r} + \frac{\varepsilon}{3}\right) < \gamma$ and $\beth_f\left(\Delta_{\acute{z}_{\widetilde{n_1}\widetilde{n_2}\widetilde{n_3}}} - \acute{z}_{\tilde{n}},\ \mathbb{r} + \frac{\varepsilon}{3}\right) < \gamma$. Then, similarly

$\sigma_t\left(\Delta_{\acute{z}_{\widetilde{n_1}\widetilde{n_2}\widetilde{n_3}}} - u, \mathbb{r} + \varepsilon\right) > 1-\gamma, \varkappa_u\left(\Delta_{\acute{z}_{\widetilde{n_1}\widetilde{n_2}\widetilde{n_3}}} - u,\ \mathbb{r} + \varepsilon\right) < \gamma$ and $\beth_f\left(\Delta_{\acute{z}_{\widetilde{n_1}\widetilde{n_2}\widetilde{n_3}}} - u,\ \mathbb{r} + \varepsilon\right) < \gamma$.

Then,

$(\widetilde{n_1}, \widetilde{n_2}, \widetilde{n_3}) \in \Big\{(n_1, n_2, n_3)$
$\in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \sigma_t \left(\Delta_{\acute{z}_{\widetilde{n_1}\widetilde{n_2}\widetilde{n_3}}} - u, \mathrm{r} + \varepsilon\right) > 1 - \gamma,$
$\varkappa_u \left(\Delta_{\acute{z}_{\widetilde{n_1}\widetilde{n_2}\widetilde{n_3}}} - u,\ \mathrm{r} + \varepsilon\right) < \gamma$ and $\beth_f \left(\Delta_{\acute{z}_{\widetilde{n_1}\widetilde{n_2}\widetilde{n_3}}} - u,\ \mathrm{r} + \varepsilon\right) < \gamma\Big\}$.

So,

$\delta_3 \left(\left\{ (n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \sigma_t \left(\Delta_{\acute{z}_{\widetilde{n_1}\widetilde{n_2}\widetilde{n_3}}} - \acute{z}_{\tilde{n}}, \mathrm{r} + \frac{\varepsilon}{3}\right) > 1 - \gamma\ ,\ \varkappa_u \left(\Delta_{\acute{z}_{\widetilde{n_1}\widetilde{n_2}\widetilde{n_3}}} - \acute{z}_{\tilde{n}},\ \mathrm{r} + \frac{\varepsilon}{3}\right) < \gamma \text{ and } \beth_f \left(\Delta_{\acute{z}_{\widetilde{n_1}\widetilde{n_2}\widetilde{n_3}}} - \acute{z}_{\tilde{n}},\ \mathrm{r} + \frac{\varepsilon}{3}\right) < \gamma\right\}\right) \leq \delta_3 \left(\left\{ (n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \sigma_t \left(\Delta_{\acute{z}_{\widetilde{n_1}\widetilde{n_2}\widetilde{n_3}}} - u, \mathrm{r} + \frac{\varepsilon}{3}\right) > 1 - \gamma\ ,\ \varkappa_u \left(\Delta_{\acute{z}_{\widetilde{n_1}\widetilde{n_2}\widetilde{n_3}}} - u,\ \mathrm{r} + \frac{\varepsilon}{3}\right) < \gamma \text{ and } \beth_f \left(\Delta_{\acute{z}_{\widetilde{n_1}\widetilde{n_2}\widetilde{n_3}}} - u,\ \mathrm{r} + \frac{\varepsilon}{3}\right) < \gamma\right\}\right)$.

Using definition of natural density

$\delta_3 \left(\left\{ (n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \sigma_t \left(\Delta_{\acute{z}_{\widetilde{n_1}\widetilde{n_2}\widetilde{n_3}}} - u, \mathrm{r} + \frac{\varepsilon}{3}\right) > 1 - \gamma\ ,\ \varkappa_u \left(\Delta_{\acute{z}_{\widetilde{n_1}\widetilde{n_2}\widetilde{n_3}}} - u,\ \mathrm{r} + \frac{\varepsilon}{3}\right) < \gamma \text{ and } \beth_f \left(\Delta_{\acute{z}_{\widetilde{n_1}\widetilde{n_2}\widetilde{n_3}}} - u,\ \mathrm{r} + \frac{\varepsilon}{3}\right) < \gamma\right\}\right) \neq 0.$

So, $u \in \mathcal{C}^{\mathrm{r}}_{\Delta_{\acute{z}_{n_1 n_2 n_3}}}$.

Theorem 3.3 Let $(\acute{z}_{n_1 n_2 n_3})$ is a triple sequences in $(\mathcal{X}, \sigma_t, \varkappa_u, \beth_f,$ $min,\ max)$ then the set $\mathcal{S}t - \mathrm{r} - LIM\left(\Delta_{\acute{z}_{n_1 n_2 n_3}}\right)$ is convex.

Proof

For $\acute{z}_1, \acute{z}_2 \in St - \mathrm{r} - LIM\left(\Delta_{\acute{z}_{n_1 n_2 n_3}}\right)$, $\varepsilon > 0$ and some $0 < \alpha < 1$, it will be shown that $((1-\alpha)y_1 + \alpha y_2) \in \mathcal{S}t - \mathrm{r} - LIM\left(\Delta_{\acute{z}_{n_1 n_2 n_3}}\right)$. Let $R, \widetilde{R}$ be defined in next form:

$$R=\Big\{(n_1,n_2,n_3)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\sigma_t\left(\Delta_{ź_{n_1n_2n_3}}-ź_1,\frac{\mathbb{r}+\varepsilon}{3(1-\alpha)}\right)$$
$$\leq 1-\gamma \quad or$$
$$\varkappa_u\left(\Delta_{ź_{n_1n_2n_3}}-ź_1,\ \frac{\mathbb{r}+\varepsilon}{3(1-\alpha)}\right)$$
$$\geq\gamma \text{ and } \beth_f\left(\Delta_{ź_{n_1n_2n_3}}-ź_1,\frac{\mathbb{r}+\varepsilon}{3(1-\alpha)}\right)\geq\gamma\Big\},$$

Then,

$$\tilde{R}=\Big\{\ (n_1,n_2,n_3)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\sigma_t\left(\ \Delta_{ź_{n_1n_2n_3}}-ź_2,\frac{\mathbb{r}+\varepsilon}{3\alpha}\right)$$
$$\leq 1-\gamma \quad or$$

$$\varkappa_u\left(\Delta_{ź_{n_1n_2n_3}}-ź_2,\ \frac{\mathbb{r}+\varepsilon}{3\alpha}\right)\geq\gamma \text{ and } \beth_f\left(\Delta_{ź_{n_1n_2n_3}}-ź_2,\frac{\mathbb{r}+\varepsilon}{3\alpha}\right)\geq\gamma\Big\}.$$

Using $ź_1,ź_2\in\mathcal{S}t-\mathbb{r}-LIM\left(\Delta_{ź_{n_1n_2n_3}}\right)$, it is written $\delta_3(R)=\delta_3(\tilde{R})=0$. Let $(\widetilde{n_1},\widetilde{n_2},\widetilde{n_3})\in R^c\cap\tilde{R}^c$, then

$$\sigma_t\left(\Delta_{ź_{n_1n_2n_3}}-\left((1-\alpha)ź_1+\alpha ź_2\right),\mathbb{r}+\varepsilon\right)$$
$$=\sigma_t\left((1-\alpha)\left(\Delta_{ź_{n_1n_2n_3}}-ź_1\right)+\alpha\left(\Delta_{ź_{n_1n_2n_3}}-ź_2\right),\mathbb{r}\right.$$
$$\left.+\,\varepsilon\right)$$

$$\geq\min\left\{\sigma_t\left((1-\alpha)\left(\Delta_{ź_{n_1n_2n_3}}-ź_1\right),\frac{\mathbb{r}+\varepsilon}{2}\right),\sigma_t\left(\alpha\left(\Delta_{ź_{n_1n_2n_3}}\right.\right.\right.$$
$$\left.\left.\left.-ź_2\right),\frac{\mathbb{r}+\varepsilon}{2}\right)\right\}$$

$$=\min\left\{\sigma_t\left(\left(\Delta_{ź_{n_1n_2n_3}}-ź_1\right),\frac{\mathbb{r}+\varepsilon}{2(1-\alpha)}\right),\sigma_t\left(\left(\Delta_{ź_{n_1n_2n_3}}-ź_2\right),\frac{\mathbb{r}+\varepsilon}{2\alpha}\right)\right\}$$
$$>1-\gamma,$$

$$\varkappa_u\left(\Delta_{ź_{n_1n_2n_3}} - \left((1-\alpha)ź_1 + \alpha ź_2\right), \mathbb{r} + \varepsilon\right)$$

$$= \varkappa_u\left((1-\alpha)\left(\Delta_{ź_{n_1n_2n_3}} - ź_1\right) + \alpha\left(\Delta_{ź_{n_1n_2n_3}} - ź_2\right), \mathbb{r} + \varepsilon\right)$$

$$\leq \max\left\{\varkappa_u\left((1-\alpha)\left(\Delta_{ź_{n_1n_2n_3}} - ź_1\right), \frac{\mathbb{r}+\varepsilon}{2}\right), \varkappa_u\left(\alpha\left(\Delta_{ź_{n_1n_2n_3}} - ź_2\right), \frac{\mathbb{r}+\varepsilon}{2}\right)\right\} < \gamma$$

and

$$\beth_f\left(\Delta_{ź_{n_1n_2n_3}} - \left((1-\alpha)ź_1 + \alpha ź_2\right), \mathbb{r} + \varepsilon\right)$$

$$= \beth_f\left((1-\alpha)\left(\Delta_{ź_{n_1n_2n_3}} - ź_1\right) + \alpha\left(\Delta_{ź_{n_1n_2n_3}} - ź_2\right), \mathbb{r} + \varepsilon\right)$$

$$\leq \max\left\{\beth_f\left(\left(\Delta_{ź_{n_1n_2n_3}} - ź_1\right), \frac{\mathbb{r}+\varepsilon}{2(1-\alpha)}\right), \beth_f\left(\alpha\left(\Delta_{ź_{n_1n_2n_3}} - ź_2\right), \frac{\mathbb{r}+\varepsilon}{2}\right)\right\} < \gamma.$$

Then, it is written that

$$\delta_3\left(\left\{(n_1,n_2,n_3) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}: \sigma_t\left(\Delta_{ź_{n_1n_2n_3}} - \left((1-\alpha)ź_1 + \alpha ź_2\right), \mathbb{r}+\varepsilon\right) \leq 1-\gamma \text{ or } \varkappa_u(ź_{n_1n_2n_3} - ((1-\alpha)ź_1 + \alpha ź_2), \mathbb{r}+\varepsilon) \geq 1-\gamma \quad \text{and} \quad \beth_f\left(\Delta_{ź_{n_1n_2n_3}} - \left((1-\alpha)ź_1 + \alpha ź_2\right), \mathbb{r}+\varepsilon\right) \geq 1-\gamma\right\}\right) = 0.$$

Consequently, $\left((1-\alpha)ź_1 + \alpha ź_2\right) \in \mathcal{S}t - \mathbb{r} - LIM\left(\Delta_{ź_{n_1n_2n_3}}\right)$. So, it is shown that the set $\mathcal{S}t - \mathbb{r} - LIM\left(\Delta_{ź_{n_1n_2n_3}}\right)$ is convex.

Theorem 3.4 Let $(\mathcal{X}, \sigma_t, \varkappa_u, ב_f,\ min,\ max)$ be a Neutrosophic Normed Spaces, $(ź_{n_1n_2n_3})$ be a triple sequences. For some $\mathbb{r} > 0$, $\gamma \in (0,1)$ and a fixed $w \in \mathcal{X}$,

$$\sigma(w,\gamma,\mathbb{r}) = \Big\{ź_{n_1n_2n_3}: \sigma_t\left(\Delta_{ź_{n_1n_2n_3}} - w, \mathbb{r}\right) > 1-\gamma, \varkappa_u\left(\Delta_{ź_{n_1n_2n_3}} - w, \mathbb{r}\right) \le \gamma,$$
$$ב_f\left(\Delta_{ź_{n_1n_2n_3}} - w, \mathbb{r}\right) \le \gamma\Big\},$$

also,

$$\overline{\sigma(w,\gamma,\mathbb{r})} = \Big\{ź_{n_1n_2n_3}: \sigma_t\left(\Delta_{ź_{n_1n_2n_3}} - w, \mathbb{r}\right) \ge 1-\gamma, \varkappa_u\left(\Delta_{ź_{n_1n_2n_3}} - w, \mathbb{r}\right)$$
$$< \gamma, ב_f\left(\Delta_{ź_{n_1n_2n_3}} - w, \mathbb{r}\right) < \gamma\Big\}.$$

Then,

$$\mathcal{C}^{\mathbb{r}}_{\Delta_{ź_{n_1n_2n_3}}} = \bigcup_{w \in \mathcal{C}_{\Delta_{ź_{n_1n_2n_3}}}} \overline{\sigma(w,\gamma,\mathbb{r})}$$

Proof

Let $ź \in \bigcup_{w\in\mathcal{C}_{ź_{n_1n_2n_3}}} \overline{\sigma(w,\gamma,\mathbb{r})}$. For some $\mathbb{r} > 0$ and given $\gamma \in (0,1)$ there exists $w \in \mathcal{C}_{ź_{n_1n_2n_3}}$ such that $\sigma_t(w - ź, \mathbb{r}) > 1-\gamma$, $\varkappa_u(w - ź, \mathbb{r}) < \gamma$ and $ב_f(w - ź, \mathbb{r}) < \gamma$.

Then, for $\bar{\mathbb{r}} > 0$ if we use $w \in \mathcal{C}_{ź_{n_1n_2n_3}}$, then there exists a set

$$P = (n_1, n_2, n_3) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}: \sigma_t\left(\Delta_{ź_{n_1n_2n_3}} - w, \bar{\mathbb{r}}\right) > 1-\gamma, \varkappa_u\left(\Delta_{ź_{n_1n_2n_3}} - w, \bar{\mathbb{r}}\right) < \gamma,$$
$$ב_f\left(\Delta_{ź_{n_1n_2n_3}} - w, \bar{\mathbb{r}}\right) < \gamma$$

and $\delta_3(P) \neq 0$. So, for $(n_1, n_2, n_3) \in P$,

$$\sigma_t\left(\Delta_{ź_{n_1n_2n_3}} - w, \mathbb{r} + \bar{\mathbb{r}}\right) \ge \min\Big\{\sigma_t\left(\Delta_{ź_{n_1n_2n_3}} - w, \bar{\mathbb{r}}\right), \sigma_t(w - ź, \mathbb{r})\Big\} > 1-\gamma,$$

$$\varkappa_u\left(\Delta_{ź_{n_1n_2n_3}} - w, ӷ + \bar{ӷ}\right) \leq \max\left\{\varkappa_u\left(\Delta_{ź_{n_1n_2n_3}} - w, \bar{ӷ}\right), \varkappa_u(w - ź, ӷ) \right\} < \gamma$$

and

$$ℶ_f\left(\Delta_{ź_{n_1n_2n_3}} - w, ӷ + \bar{ӷ}\right) \leq \max\left\{ℶ_f\left(\Delta_{ź_{n_1n_2n_3}} - w, \bar{ӷ}\right), ℶ_f(w - ź, ӷ) \right\} < \gamma.$$

From hence,

$$\delta_3\left\{\left((n_1, n_2, n_3) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}: \sigma_t\left(\Delta_{ź_{n_1n_2n_3}} - ź, ӷ + \bar{ӷ}\right) > 1 - \gamma, \varkappa_u\left(\Delta_{ź_{n_1n_2n_3}} - ź, ӷ + \bar{ӷ}\right) < \gamma \text{ and } ℶ_f\left(\Delta_{ź_{n_1n_2n_3}} - ź, ӷ + \bar{ӷ}\right) < \gamma\right)\right\} \neq 0.$$

Thus, $ź \in \mathcal{C}^{ӷ}_{ź_{n_1n_2n_3}}$ and then $\bigcup_{w\in\mathcal{C}_{\Delta_{ź_{n_1n_2n_3}}}} \overline{\sigma(w, \gamma, ӷ)} \subseteq \mathcal{C}^{ӷ}_{ź_{n_1n_2n_3}}$.

It is easily demonstrated from definition that $\mathcal{C}^{ӷ}_{ź_{n_1n_2n_3}} \subseteq \bigcup_{w\in\mathcal{C}_{\Delta_{ź_{n_1n_2n_3}}}} \overline{\sigma(w, \gamma, ӷ)}$

Theorem 3.5

Let $(ź_{n_1n_2n_3})$ be a triple sequences in $(\mathcal{X}, \sigma_t, \varkappa_u, ℶ_f, \ min, \ max)$ and $(ź_{n_1n_2n_3})$ is statistically convergent to $ź$. For some $ӷ > 0$, there exists $\gamma \in (0,1)$:

$$\mathcal{St} - ӷ - LIM\left(\Delta_{ź_{n_1n_2n_3}}\right) = \overline{\sigma(y, \gamma, ӷ)}.$$

Proof

For $\ddot{r} > 0$, if we use statistical convergence of $(ź_{n_1n_2n_3})$, there exists

$$T = \{(n_1, n_2, n_3) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}: \sigma_t\left(\Delta_{ź_{n_1n_2n_3}} - ź, \ddot{ӷ}\right) \leq 1 - \gamma \ or\ \varkappa_u\left(\Delta_{ź_{n_1n_2n_3}} - ź, \ddot{ӷ}\right) \geq \gamma$$

$$and\ ℶ_f\left(\Delta_{ź_{n_1n_2n_3}} - ź, \ddot{ӷ}) \geq \gamma\right\} \text{ and } \delta_3(T) = 0.$$

Let $s \in \overline{\sigma(ź, \gamma, ӷ)}$ and for $(n_1, n_2, n_3) \in T^c$, then

$\sigma_t\left(\Delta_{\acute{z}_{n_1n_2n_3}} - s, \mathbb{r} + \ddot{\mathbb{r}}\right) > 1 - \gamma, \varkappa_u\left(\Delta_{\acute{z}_{n_1n_2n_3}} - s, \mathbb{r} + \ddot{\mathbb{r}}\right) < \gamma$ and $\beth_f\left(\Delta_{\acute{z}_{n_1n_2n_3}} - s, \mathbb{r} + \ddot{\mathbb{r}}\right) < \gamma$.

That is $s \in \mathcal{S}t - \mathbb{r} - LIM\left(\Delta_{\acute{z}_{n_1n_2n_3}}\right)$. So, $\overline{\sigma(\acute{z}, \gamma, \mathbb{r})} \subseteq \mathcal{S}t - \mathbb{r} - LIM\left(\Delta_{\acute{z}_{n_1n_2n_3}}\right)$. Furthermore, $\mathcal{S}t - \mathbb{r} - LIM\left(\Delta_{\acute{z}_{n_1n_2n_3}}\right) \subseteq \overline{\sigma(\acute{z}, \gamma, \mathbb{r})}$. Consequently, $\mathcal{S}t - \mathbb{r} - LIM\left(\Delta_{\acute{z}_{n_1n_2n_3}}\right) = \overline{\sigma(\acute{z}, \gamma, \mathbb{r})}$.

4. CONCLUSION

Rough convergence is an important concept for the fields of fuzzy, intuitionistic fuzzy and neutrosophic theory. We carry forward the notion of coarse statistical convergence defined on Neutrosophistic normed spaces using triple difference sequences. Thus, we extend some important well-known results. Important topological properties of the set of coarse statistical limit points are given.

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REFERENCES

Adhya, S. & Deb Ray, A. (2022). On weak G-completeness for fuzzy metric spaces. Soft Computing, 1-7.

Akcay, F. G., & Aytar, S. (2015). Rough convergence of a sequence of fuzzy numbers. *Bull. Math. Anal. Appl*, *7*(4), 17-23.

Antal, R. Chawla, M. & Kumar, V. (2021). Rough statistical convergence in intuitionistic fuzzy normed spaces. Filomat, 35(13), 4405-4416.

Atanassov, K. (1986) Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20(1), pp.87–96.

Aytar, S. (2008), Rough statistical convergence, Numer. Funct. Anal. Optimiz. 29(3-4) 291–303.

Bilgin, N. G. & Bozma, G. (2020). On Fuzzy n-Normed Spaces Lacunary Statistical Convergence of order α . i-Manager's Journal on Mathematics, 9(2), 1.

Bilgin, N. G. & Bozma, G. (2021). Fibonacci Lacunary Statistical Convergence of order γ in IFNLS.

Bilgin, N. G. (2022). Rough statistical convergence in neutrosophic normed spaces. *Euroasia Journal of Mathematics, Engineering, Natural & Medical Sciences*, *9*(21), 47-55.

Bilgin, N. G., (2022). New Kinds of Statistical Convergence on Neutrosophic Normed Spaces, Math Sciences in International Research, Egitim Publishing.

Chandra Das, P. (2018). Fuzzy normed linear sequence space. Proyecciones (Antofagasta), 37(2), 389-403.

Coskun, E., (2000), Systems on intuitionistic fuzzy special sets and intuitionistic fuzzy special measures, Information Sciences, 128, 105-118.

Debnath, S. & Subramanian, N. (2017). Rough statistical convergence on triple sequences. Proyecciones (Antofagasta), 36(4), 685-699.

Granados, C. & Dhital, A. (2021). Statistical convergence of double sequences in neutrosophic normed spaces. Neutrosophic Sets and Systems, 42, 333-344.

Granados, C. & Das, S. (2022). On (λ, μ, ζ)-Zweier ideal convergence in intuitionistic fuzzy normed spaces. Yugoslav Journal of Operations Research.

Kausar R, Farid HMA, Riaz M, Gonul Bilgin N. Innovative CODAS Algorithm for q-Rung Orthopair Fuzzy Information and Cancer Risk Assessment. *Symmetry*. 2023; 15(1):205.

Kirisci M. and Simsek N., (2020) Neutrosophic normed spaces and statistical convergence,The Journal of Analysis, 1-15.

Kisi. O. & Gurdal, V. (2022). On Triple Difference Sequences of Real Numbers in Neutrosophic Normed Spaces. Communications in Advanced Mathematical Sciences, 5(1), 35-45.

Kisi, O. (2021), Ideal convergence of sequences in neutrosophic normed spaces. Journal of Intelligent & Fuzzy Systems, (Preprint), 1-10.

Phu H.X., (2001), Rough convergence in normed linear spaces, Numer. Funct. Anal. Optim. 22(1-2) 199–222.

Smarandache, F. Neutrosophy, Neutrosophic Probability, Set, and Logic, ProQuest Information and Learning. Ann Arbor, Michigan, USA, 1998.

Sahin, M. & Kargın, A. (2017). Neutrosophic triplet normed space. Open Physics, 15(1), 697-704.

Sahiner, A. Gurdal, M. & Duden, F. K. (2007). Triple sequences and their statistical convergence. Selcuk J. Appl. Math. 8(2), 49-55.

Zadeh, L.A. (1965) Fuzzy sets, Information and Control, Vol. 8, No. 3, pp.338–356.

Olmez, O., & Aytar, S. (2021). On the rough hausdorff convergence. *Sigma: Journal of Engineering & Natural Sciences,39*.

Malik, H. M., & Akram, M. (2018). A new approach based on intuitionistic fuzzy rough graphs for decision-making. *Journal of Intelligent & Fuzzy Systems*, *34*(4), 2325-2342.

Demir, N. & Gumus, H. (2022). Rough statistical convergence for difference sequences. *Kragujevac Journal of Mathematics*, *46*(5),